Histories and observables in covariant field theory

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Abstract

Motivated by DeWitt's viewpoint of covariant field theory, we define a general notion of non-local classical observable that applies to many physical Lagrangian systems (with bosonic and fermionic variables), by using methods that are now standard in algebraic geometry. We review the methods of local functional calculus, as they are presented by Beilinson and Drinfeld, and relate them to our construction. We partially explain the relation of these with Vinogradov's secondary calculus. The methods present here are all necessary to understand mathematically properly and with simple notions the full renormalization of the standard model, based on functional integral methods. Our approach is close in spirit to non-perturbative methods since we work with actual functions on spaces of fields, and not only formal power series. This article can be seen as an introduction to well-grounded classical physical mathematics, and as a good starting point to study quantum physical mathematics, which make frequent use of non-local functionals, like for example in the computation of Wilson's effective action. We finish by describing briefly a coordinate-free approach to the classical Batalin-Vilkovisky formalism for general gauge theories, in the language of homotopical geometry.

Contents

1 Introduction

We [physicists] often do not know a priori
just where a given formalism [...] will take us.
We are compelled to leave it to the mathematicians
to tidy things up after we have left the playing field.
Bryce S. DeWitt.
The global approach to quantum field theory.

A recurrent difficulty that the average mathematician has to face if he opens an experimental physics book is the scarcity of definitions for the mathematical objects used in computations. This does not mean that the computations of experimental physicists are false, only that they are hard to approach from a mathematician's viewpoint.

The aim of this article is to show how standard abstract methods of pure mathematics (functors, sheaves, homotopical spaces, monoidal categories and operads) can be seen as parts of applied mathematics, since they are all necessary to explain to a mathematician the standard model of particle physics or supersymmetric field theory without coordinates. The main advantage of computations without coordinates is that they are often very algebraic, and general enough to be used in very different contexts. Moreover, they open the road to many interesting mathematical applications.

The starting point of particle physics is the study of variational problems, their symmetries and conservation laws. There are at least three ways to study such problems: functional analysis, local functional calculus and non-local functional calculus.

The functional analytic methods (analysis on spaces of functions) proved to be very useful in the linear case but have a quite limited scope for non linear problems.

Local functional calculus on jet spaces has been developed by many authors, starting from Noether and her famous theorems (see [KS06b]), and including Gelfand, Manin, Vinogradov and Gromov. We will mainly be interested in Vinogradov's $\mathcal{C}$-spectral sequence [Vin01] and secondary calculus, studied in the smooth setting in various reference books by many people (see [Vin01], [BCD$^+$99], [KV98]), and the algebraic approach of Beilinson and Drinfeld [BD04], using the language of differential algebra, which originated in Ritt's school [Rit66]. To sum up, these methods allow one to compute symmetries and conservation laws of partial differential equations systematically and to solve the inverse problem of variational calculus algebraically. It also allows one to prove deep results on partial differential equations like the h-principle [Gro86], but we will not go into this since we are only interested in the very formal computations of physicists.

Non-local functional calculus is all around in the physical literature, and its mathematical formalization is very close to Grothendieck's functorial approach of geometry [AGV73]. It was first introduced in physics in a special case by Souriau [Sou97] and his school [IZ99] (see also [DF99]). It will prove very

close to the physicists' way of thinking of variational problems. This functorial approach is already used in finite dimension in the IAS lectures [DM99], and for spaces of fields in Lott's article [Lot90], but was not systematically developed mathematically there (no sheaf condition, for example).

These three methods have their advantages and drawbacks. Since the literature on functional methods is already very large, we will concentrate on the two other approaches. One can compare these two last theories to Grothendieck's scheme theory in the setting of partial differential equations: it does not allow us to solve the equations explicitly (this can be done, anyway, only in very special cases), but it gives powerful methods to define and compute some very interesting invariants associated to the geometry of the space of solutions. These methods are based on differential calculus on the "space of solutions" of the given partial differential equations. The local and non-local calculus give two useful definitions of this notion of "space of solution", which is central in modern physics.

The algebro-geometric methods are based on the "punctual" approach to geometry, which is essentially the way physicists think: one does not have to wonder exactly on which functional space one works, the only important things being what a parameterized function is, and the formal changes of parameterizations that are allowed (i.e., the categories and morphisms in play). These methods already proved to be very useful in understanding fermionic differential and integral calculus and they give a geometrically intuitive way to work with spaces of solutions of nonlinear partial differential equations. Our methods are aimed at the study of non-perturbative quantum field theory, but here we describe here mainly classical field theory.

We finish by briefly describing a general coordinate-free approach to the classical Batalin-Vilkovisky formalism for general gauge theories.

In this paper, all classical manifolds used are implicitly supposed to be oriented, if one needs to integrate differential forms on them.

2 Points, coordinates and histories in geometry and physics

There are two complementary ways of studying spaces in geometry and physics.

1. The *functional viewpoint*, based on the notion of a *coordinate* function, also called observable (see Nestruev [Nes03], Connes [Con94] and most of the literature on mathematical physics), translates most of the geometrical constructions in algebraic terms.

2. The *punctual viewpoint*, based on the notion of a *point* (see [Sou97], [IZ99] and [Gro60a]), studies a given space by giving all its parameterized families of points with values in some given standard building blocks, like open subsets of $\mathbb{R}^n$ for example.

Both of these methods have their advantages. The main objects of modern physics, called spaces of histories or spaces of fields, are functional spaces. In studying such infinite-dimensional spaces, the functional viewpoint becomes sometimes too cumbersome or even inefficient, and the systematic use of the punctual viewpoint proves to be closer to the physicists' (sometimes informal) language. Various combinations of both of them will give optimal geometrical contexts for physics.

2.1 Lagrangian variational problems

We first give a definition of a lagrangian variational problem, which is general enough to treat many variational problems that appear in classical and quantum physics (classical mechanics, Yang-Mills theory, general relativity, fermionic field theory and supersymmetric sigma models). We base our approach on a general notion of space, which will be described in this paper.

Definition 1. A lagrangian variational problem is composed of the following data:

1. a space M called the parameter space for trajectories,
2. a space C called the configuration space for trajectories,
3. a morphism $\pi : C \to M$ (often supposed to be surjective),
4. a subspace $H \subset \Gamma(M, C)$ of the space of sections of π

   $$\Gamma(M, C) := \{x : M \to C,\ \pi \circ x = \mathrm{id}\},$$

   called the space of histories, and
5. a functional $S : H \to A$ (where A is a space in rings that is often the real line $\mathbb{R}$ or $\mathbb{R}[[\hbar]]$) called the action functional.

The space of classical trajectories for the variational problem is the subspace T of H defined by

$$T = \{x \in H |\ d_x S = 0\}.$$

If B is another space, a classical B-valued observable is a functional $F : T \to B$ and a quantum B-valued observable is a functional $F : H \to B$.

We will now give some physical examples, without going into details. The definitions of the types of space that are necessary to formalize these examples properly in the above language arise with an increasing level of difficulties.

In classical mechanics, the parameter space M for trajectories is a compact time interval, e.g. $M = [0, 1]$, the configuration bundle is the natural projection $\pi : C = [0, 1] \times \mathbb{R}^3 \to [0, 1] = M$, and the space of histories is the space of trajectories $x : [0, 1] \to \mathbb{R}^3$ with some fixed starting and ending points x_0 and x_1. The action functional of the free particle is given by the formula

$$S(x) = \int_M \frac{1}{2} m \|\partial_t x\|^2 dt.$$

To describe the variational problem of pure Yang-Mills theory, one needs a lorentzian spacetime manifold (M, g) and a principal G-bundle P on M. The projection $\pi : C = \mathrm{Conn}_G(P) \to M$ is the bundle whose sections A are principal G-connections (G-equivariant covariant derivatives). The action functional is given by the formula

$$S(A) = \int_M F_A \wedge *F_A$$

where F_A is the curvature of the connection A on P.

In fermionic field theory, one starts as before with a four-dimensional lorentzian manifold (M, g) and a spinor bundle $S \to M$. The projection $\pi : C = \Pi S \to M$ is the odd fiber bundle associated to S (with fiber odd supervector spaces) and the action functional is given by

$$S(\psi) = \int_M \langle \psi, \not{D}\psi \rangle dx$$

where the above pairing is described in [DM99]. It is already hard there to give a proper sense to this expression, because a usual section $\psi : M \to \Pi S$ is essentially trivial.

In supersymmetric sigma models, one starts with a supervariety (see below for a general definition), for example $\mathbb{R}^{1|1}$, and works with the bundle $\pi : C = \mathbb{R} \times \mathbb{R}^{1|1} \to \mathbb{R}^{1|1}$. The supersymmetric lagrangian is then given by a superintegral over $\mathbb{R}^{1|1}$.

To be able to treat all these examples on an equal footing, we will need a flexible enough notion of "space" of trajectories and histories.

2.2 Points and coordinates: useful nonsense

To define a very general notion of space, one needs a category of geometrical building blocks, which we call Legos. This must be equipped with a (subcanonical) Grothendieck topology τ.

Recall that if Legos is a category, and if we denote $\text{Legos}^\vee$ the category of contravariant functors from Legos to Sets, there is a fully faithful Yoneda embedding

$$\begin{array}{ccc} \text{Legos} & \to & \text{Legos}^\vee \\ X & \mapsto & \underline{X} := \mathrm{Hom}(., X) : \text{Legos} \to \text{Sets}. \end{array}$$

This gives an embedding of Legos in a category that contains all limits and colimits.

If one starts with the opposite category Legos^{op} instead of Legos, the ioneda embedding will be given by

$$\begin{array}{ccc} \text{Legos}^{op} & \to & (\text{Legos}^{op})^\vee \\ X & \mapsto & \underline{X} := \mathrm{Hom}_{\text{Legos}^{op}}(., X) = \mathrm{Hom}_{\text{Legos}}(X, .). \end{array}$$

The above constructions are the categorical counterparts of the two viewpoint of spaces used by physicists:

1. in the punctual viewpoint, points of a space X with values in legos are given by the contravariant functor $\mathrm{Hom}(.,X) : \textsc{Legos} \to \textsc{Sets}$;

2. in the functional viewpoint, coordinates on a space X are given by the covariant functor $\mathrm{Hom}(X,.) : \textsc{Legos} \to \textsc{Sets}$, and as seen above, these are also points of X with values in objects of $\textsc{Legos}^{op}$.

Here are the main advantages of the two approaches.

1. The punctual viewpoint is very natural for the study of contravariant functorial constructions (i.e. constructions with a pull-back), like differential forms for example.

2. The functional viewpoint is very natural for the study of covariant constructions like vector fields and differential operators.

It will thus often be useful to combine those two approaches by using as Legos categories of algebras of functions, for which one can apply covariant and contravariant constructions.

If one works with the category $\textsc{Legos}^{\vee}$, one gets into trouble when pasting building blocks. Indeed, suppose for example that $\textsc{Legos} = \mathrm{Open}$ is the category of open subsets of $\mathbb{R}^n$ for varying n. Then if an open set U is covered by two open subsets, i.e.,

$$U = U_1 \cup U_2 =: U_1 \coprod_U U_2,$$

one usually does not have

$$\underline{U} = \underline{U_1} \coprod_{\underline{U}} \underline{U_2}$$

in the category $\textsc{Legos}^{\vee}$. One will thus get into trouble if one wants do define varieties by pasting the spaces associated to legos. The sheaf condition is here exactly to prevent this bad situation happening.

Spaces will thus simply be the category $\mathrm{Sh}(\textsc{Legos}, \tau) \subset \textsc{Legos}^{\vee}$ of sheaves of sets on Legos with respect to the topology τ. Yoneda's lemma implies that the canonical functor

$$\begin{array}{ccc} \textsc{Legos} & \to & \mathrm{Sh}(\textsc{Legos}, \tau) \\ U & \mapsto & [\underline{U} := \mathrm{Hom}(., U) : V \mapsto \mathrm{Hom}(V, U)] \end{array}$$

is an embedding. A common denomination for a sheaf $F \in \mathrm{Sh}(\textsc{Legos}, \tau)$ is that of the functor of points of the corresponding space with values in Legos.

This thus gives a definition of spaces by their points.

In usual (finite-dimensional) geometry, one defines a particular class of spaces, called geometrical sheaves. These spaces are covered by a special kind of morphism

$$\coprod U_i \to F$$

from a union of legos. The precise definition of such geometric contexts can be found in [TV08a]. We do not give it in details because it does not apply to (usually infinite-dimensional) spaces of maps, which are the central objects of covariant field theory.

We now recall the general definitions from [KS06a].

Definition 2. Let Legos be a category. A Grothendieck topology τ on Legos is the datum of families of morphisms $\{f_i : U_i \to U\}$, called covering families, and denoted Cov_U, such that the following holds:

1. (Identity) The identity map $\{\mathrm{id} : U \to U\}$ belongs to Cov_U.
2. (Refinement) If $\{f_i : U_i \to U\}$ belongs to Cov_U and $\{g_{i,j} : U_{i,j} \to U_i\}$ belong to Cov_{U_i}, then the composed covering family $\{f_i \circ g_{i,j} : U_{i,j} \to U\}$ belongs to Cov_U.
3. (Base change) If $\{f_i : U_i \to U\}$ belongs to Cov_U and $f : V \to U$ is a morphism, then $\{f_i \times_U f : U_i \times_U V \to V\}$ belongs to Cov_V.
4. (Local nature) If $\{f_i : U_i \to U\}$ belongs to Cov_U and $\{f_j : V_j \to U\}$ is a small family of arbitrary morphisms such that $f_i \times_U f_j : U_i \times_U V_j \to U_i$ belongs to Cov_{U_i}, then $\{f_j\}$ belongs to Cov_U.

A category Legos equipped with a Grothendieck topology τ is called a site.

We remark that for this definition to make sense one needs the fiber products that appear in it to exist in the given category. A more flexible and general definition in terms of sieves can be found in [KS06a].

Suppose that we work on the category $\textsc{Legos} = \mathrm{Open}_X$ of open subsets of a given topological space X, with inclusion morphisms and its usual topology. The base change axiom then says that a covering of $U \subset X$ induces a covering of its open subsets. The local character means that families of coverings of elements of a given covering induce a (refined) covering.

Definition 3. Let $(\textsc{Legos}, \tau)$ be a category with Grothendieck topology. A functor $X : \textsc{Legos}^{op} \to \textsc{Sets}$ is called a sheaf if the sequence

$$X(U) \longrightarrow \prod_i X(U_i) \rightrightarrows \prod_{i,j} X(U_i \times_U U_j)$$

is exact. The category of sheaves is denoted $\mathrm{Sh}(\textsc{Legos}, \tau)$.

One can think of a sheaf in this sense as something analogous to the continuous functions on a topological space. A continuous function on an open set is uniquely defined by a family of continuous functions on the open subsets of a given covering, whose values are equal on their intersections.

From now on, one further supposes that, for all legos U, $\underline{U}$ is a sheaf for the given topology.

The main advantage of the category of sheaves is the following classical fact.

Lemma 1. *Let* Open *be the category of open subsets of* $\mathbb{R}^n$ *for varying* n *with its usual topology. Let* $U = U_1 \coprod_U U_2$ *be a covering of an open subset by two open subsets. Then one has*

$$\underline{U} = \underline{U_1} \coprod_{\underline{U}} \underline{U_2}$$

if the coproduct is taken in the category of sheaves on Open.

The most standard example of a punctual geometrical setting is given by the theory of diffeology, which was developed by Smith [Smi66] and Chen [Che77], and used in the physical setting by Souriau (see [Sou97] and [IZ99] for references and historical background, and [BH08] for an overview) to explain the geometric methods used by physicists to study variational problems.

Definition 4. Let $\mathrm{Open}_{\mathcal{C}^\infty}$ be the category of open subsets of $\mathbb{R}^n$ for varying n with smooth maps between them and τ be the usual topology on open subsets. The category of diffeologies (also called smooth spaces) is the category $\mathrm{Sh}(\mathrm{Open}_{\mathcal{C}^\infty}, \tau)$.

There is a fully faithful embedding of the category of smooth varieties in diffeologies that sends a variety X to its functor of parameterized points

$$\begin{array}{rccc} \underline{X}: & \mathrm{Open}_{\mathcal{C}^\infty}^{op} & \to & \textsc{Sets} \\ & U & \mapsto & \mathrm{Hom}_{\mathcal{C}^\infty}(U, X). \end{array}$$

If we replace $\mathrm{Open}_{\mathcal{C}^\infty}$ by its opposite category $\mathrm{Open}_{\mathcal{C}^\infty}^{op}$, we get the functional viewpoint of varieties: there is a natural embedding of the category of smooth varieties in the category of covariant functors from $\mathrm{Open}_{\mathcal{C}^\infty}$ to Sets given by sending X to $\mathrm{Hom}(X, .)$. Since every open subset U of $\mathbb{R}^n$ can be thought as given with an embedding $U \subset \mathbb{R}^n$, we always have

$$\mathrm{Hom}(X, U) \subset \mathrm{Hom}(X, \mathbb{R}^n) = \mathrm{Hom}(X, \mathbb{R})^n.$$

This simple result implies that from the functional viewpoint, it is often enough to consider functions with values in $\mathbb{R}$, and this is actually what analysts usually do, and they are right!

We now come to the definition of contravariant constructions on spaces.

Definition 5. Let $(\textsc{Legos}, \tau)$ be a category with Grothendieck topology and let X be a space (i.e., a sheaf for the given topology). Let C be a given category (of types of structure... think of vector spaces) and let $\Omega : \textsc{Legos}^{op} \to C$ be a contravariant functor (given by a standard contravariant structure on Legos... think of differential forms) that is moreover a sheaf. If $\textsc{Legos}/X$ denotes the category of morphisms $U \to X$ for $U \in \textsc{Legos}$, one defines Ω_X as the functor $\Omega_X : \textsc{Legos}/X \to C$. An element of Ω_X is defined as a family of elements $\omega_U \in \Omega_X(U)$ compatible with the functorial maps $\Omega_{U,V} : \Omega_X(U) \to \Omega_X(V)$ for $f : V \to U$ a morphism in $\textsc{Legos}/X$.

As an example, in the diffeological setting, the differential graded algebra of differential form $U \mapsto (\Omega^*(U), d)$ on LEGOS has a well-defined pull-back along smooth morphisms $f : U \to V$. This thus gives a definition of the notion of differential form and a Rham complex on a diffeology.

Definition 6. Let $X : (\mathrm{Open}^{op}_{\mathcal{C}^\infty}, \tau) \to$ SETS be a diffeological space. A differential form on X is the datum, for every morphism $x : U \to X$ of a differential form $x^*\omega$, such that for every $f : U \to V$ and $x' : V \to X$ such that $x' \circ f = x$, one has

$$f^*((x')^*\omega) = x^*\omega.$$

More generally, one can define the notion of a bundle over a diffeological space X in a similar way by using the "functor"

$$\Omega = \mathrm{Bun} : \mathrm{Open}^{op}_{\mathcal{C}^\infty} \to \mathrm{Groupoids}$$

that sends an open U to the groupoid of bundles on U. Let $X : (\mathrm{Open}^{op}_{\mathcal{C}^\infty}, \tau) \to$ SETS be a diffeological space. A bundle on X is (roughly speaking) the datum, for every morphism $x : U \to X$ of a bundle x^*E on U, and for every $f : U \to V$ and $x' : V \to X$ such that $x' \circ f = x$, of an isomorphism

$$f^*((x')^*E) \cong x^*E,$$

with an additional identity associated to pairs of morphisms between points. The proper mathematical formulation of this construction involves homotopical methods, which we will talk about latter.

Remark 1. If we replace LEGOS by LEGOS^{op}, and equip it with a Grothendieck topology τ' (also called a Grothendieck cotopology on LEGOS), a (LEGOS$^{op}, \tau'$)-space will be defined by its spaces of functions $\mathrm{Hom}(X, .)$: LEGOS $\to$ SETS that must fulfil a cosheaf property to preserve finite limits of LEGOS. In this setting, we get natural definitions of covariant operations like vector fields or differential operators.

2.3 Equations and their solution functor with values in algebras

Another take at the functorial approach to geometry is by the study of equations and their solution functor (also called functor of points), as explained in the introduction of the Springer version of EGA [Gro60b]. Most of the spaces used in physics are described by some equations.

Let $\{P_i(x_1, \dots, x_n)\}$ be a family of polynomials with real coefficients. To write down the equations $P_i = 0$, one only needs a commutative $\mathbb{R}$-algebra. If A is a commutative $\mathbb{R}$-algebra, one can look for the solutions of $P_i = 0$ in A^n. This gives a functor

$$\mathrm{Sol}_{P_i=0} : \mathrm{ALG}_{\mathbb{R}} \to \mathrm{SETS}$$

defined by

$$\mathrm{Sol}_{P_i=0}(A) := \{(x_1, \dots, x_n) \in A^n, \forall i,\ P_i(x_1, \dots, x_n) = 0\}.$$

The universal properties of the polynomial and of the quotient ring essentially tell us that this functor is isomorphic to the functor

$$\begin{array}{rccc} \underline{\mathrm{Spec}}(\mathbb{R}[x_1,\ldots,x_n]/(P_i)) : & \textsc{Alg}_{\mathbb{R}} & \to & \textsc{Sets} \\ & A & \mapsto & \mathrm{Hom}_{\textsc{Alg}_{\mathbb{R}}}(\mathbb{R}[x_1,\ldots,x_n]/(P_i), A) \end{array}$$

corresponding to the algebra $\mathbb{R}[x_1,\ldots,x_n]/(P_i)$. The localization maps $A \to A[f^{-1}]$ for $f \in A$ define a Grothendieck topology τ_{Zar} on $\textsc{Legos} = \textsc{Alg}_{\mathbb{R}}^{op}$ called the Zariski topology. The functor $\mathrm{Sol}_{P=0}$ is a sheaf for this topology.

Definition 7. An algebraic space over $\mathbb{R}$ is a sheaf on the site $(\textsc{Alg}_{\mathbb{R}}^{op}, \tau_{Zar})$, i.e., a covariant functor

$$X : \textsc{Alg}_{\mathbb{R}} \to \textsc{Sets}$$

that is a sheaf fo τ_{Zar}.

A scheme over $\mathbb{R}$ (of finite type) is essentially an algebraic space that "can be covered" (for more details, see [TV08a]) by some solution spaces.

If one wants to work with equations defined by smooth functions, one has to refine the Legos category to a particular kind of algebra. For A a real algebra and $a \in A$, we denote $\mathrm{Spec}_{\mathbb{R}}(A) := \mathrm{Hom}_{\textsc{Alg}_{\mathbb{R}}}(A, \mathbb{R})$, and

$$\begin{array}{rccc} \mathrm{ev}_a : & \mathrm{Spec}_{\mathbb{R}}(A) & \to & \mathbb{R} \\ & x & \mapsto & x(a). \end{array}$$

Definition 8. An algebra A is called

1. smoothly affine if for every n, the natural map

$$\begin{array}{ccc} \underline{\mathrm{Spec}}(\mathcal{C}^\infty(\mathbb{R}^n))(A) := \mathrm{Hom}_{\textsc{Alg}_{\mathbb{R}}}(\mathcal{C}^\infty(\mathbb{R}^n), A) & \to & A^n \\ \varphi & \mapsto & \varphi(x_1),\ldots,\varphi(x_n) \end{array}$$

   is bijective;

2. smoothly closed geometric if for every $a_1,\ldots,a_n \in A^n$ and $f \in \mathcal{C}^\infty(\mathbb{R}^n)$, there exists a unique $a \in A$ such that the function

$$f \circ (\mathrm{ev}_{a_1} \times \cdots \times \mathrm{ev}_{a_n}) : \mathrm{Spec}_{\mathbb{R}}(A) \to \mathbb{R}$$

   equals ev_a.

We denote $\textsc{Alg}_{sa}$ (respectively, $\textsc{Alg}_{scg}$) the category of smoothly affine (respectively, smoothly closed geometric) real algebras.

The notion of smoothly closed geometric algebra was first introduced by Nestruev [Nes03].

The main advantage of smoothly affine algebras over usual algebras is that they allow us to make sense of the solution space to smooth equations. If $\{f_i : \mathbb{R}^n \to \mathbb{R}\}$ is a family of smooth functions on $\mathbb{R}^n$, one can look for the solution to $f_i(x_1,\ldots,x_n) = 0$ for $(x_1,\ldots,x_n) \in A^n$ if A is in $\textsc{Alg}_{sa}$. Indeed,

$(x_1, \ldots, x_n)$ corresponds to a morphism $\varphi_{x_1,\ldots,x_n} : \mathcal{C}^\infty(\mathbb{R}^n) \to A$ and one can evaluate it at the f_i to get elements $\varphi_{x_1,\ldots,x_n}(f_i) \in A$. This gives a functor

$$\begin{array}{rccc} \mathrm{Sol}_{f_i=0}: & \textsc{Alg}_{sa} & \to & \textsc{Sets} \\ & A & \mapsto & \{(x_1,\ldots,x_n) \in A^n,\ \varphi_{x_1,\ldots,x_n}(f_i) = 0\}. \end{array}$$

It is then easy to show that one has a functor isomorphism

$$\mathrm{Sol}_{f_i=0} \cong \underline{\mathrm{Spec}}(\mathcal{C}^\infty(\mathbb{R}^n)/(f_i)) : \textsc{Alg}_{sa} \to \textsc{Sets}$$

(we carefully inform the reader that $\mathcal{C}^\infty(\mathbb{R}^n)/(f_i)$ is not smoothly affine in general).

Theorem 1. *1. An algebra that is smoothly closed geometric is smoothly affine.*

2. *Let $\otimes_{sa}$ be the tensor product in $\textsc{Alg}_{sa}$. Then*

$$\mathcal{C}^\infty(\mathbb{R}^p) \otimes_{sa} \mathcal{C}^\infty(\mathbb{R}^q) \cong \mathcal{C}^\infty(\mathbb{R}^{p+q}).$$

3. *If $U \subset \mathbb{R}^n$ is an open subset then $\mathcal{C}^\infty(U)$ is smoothly closed and geometric.*

4. *The functor*

$$\begin{array}{rccc} \mathcal{C}^\infty: & \mathrm{Open}_{\mathcal{C}^\infty} & \to & \textsc{Alg}_{sa} \\ & U & \mapsto & \mathcal{C}^\infty(U) \end{array}$$

is fully faithful with essential image denoted $\textsc{Alg}_{\mathcal{C}^\infty}$. It induces an equivalence of sites between $(\mathrm{Open}_{\mathcal{C}^\infty}, \tau)$ and $(\textsc{Alg}^{op}_{\mathcal{C}^\infty}, \tau_{Zar})$.

5. *If X is a smooth variety, then $\mathcal{C}^\infty(X)$ is smoothly closed and geometric.*

6. *If $\otimes_{scg}$ denotes the tensor product in $\textsc{Alg}_{scg}$ and M and N are smooth varieties, then*

$$\mathcal{C}^\infty(M) \otimes_{scg} \mathcal{C}^\infty(N) \cong \mathcal{C}^\infty(M \times N).$$

7. *There is a natural adjoint $\textsc{Alg}_{\mathbb{R}} \to \textsc{Alg}_{scg}$ to the forgetful functor $\textsc{Alg}_{scg} \to \textsc{Alg}_{\mathbb{R}}$. It is called the smooth geometric closure, and is denoted $A \mapsto \overline{A}^{scg}$.*

Proof. Let A be a smoothly closed geometric algebra. Let $(a_1, \ldots, a_n) \in A^n$ be a family of elements in A^n and $f \in \mathcal{C}^\infty(\mathbb{R}^n)$ be a smooth function. By definition, there exists a unique $a_f \in A$ such that

$$f \circ (\mathrm{ev}_{a_1} \times \cdots \times \mathrm{ev}_{a_n}) = \mathrm{ev}_a : \mathrm{Spec}_{\mathbb{R}}(A) \to \mathbb{R}.$$

The map

$$\begin{array}{rccc} \varphi_{a_1,\ldots,a_n}: & \mathcal{C}^\infty(\mathbb{R}^n) & \to & A \\ & f & \mapsto & a_f \end{array}$$

is an algebra morphism by construction, and it is uniquely defined by $a_1, \ldots, a_n$. This gives the fact that the natural map

$$\mathrm{Hom}(\mathcal{C}^\infty(\mathbb{R}^n), A) \to A^n$$

is bijective, so that A is smoothly affine. By definition of the tensor product, if A is smoothly affine, one has a bijection

$$\begin{array}{rcl} \mathrm{Hom}_{\mathrm{ALG}_{sa}}(\mathcal{C}^\infty(\mathbb{R}^p) \otimes_{sa} \mathcal{C}^\infty(\mathbb{R}^q), A) & \cong & \mathrm{Hom}(\mathcal{C}^\infty(\mathbb{R}^p), A) \times \mathrm{Hom}(\mathcal{C}^\infty(\mathbb{R}^q), A) \\ & = & A^p \times A^q. \end{array}$$

But since A is smoothly affine, one gets

$$A^{p+q} \cong \mathrm{Hom}_{\mathrm{ALG}_{sa}}(\mathcal{C}^\infty(\mathbb{R}^{p+q}), A).$$

This shows that $\mathcal{C}^\infty(\mathbb{R}^p) \otimes_{sa} \mathcal{C}^\infty(\mathbb{R}^q)$ and $\mathcal{C}^\infty(\mathbb{R}^{p+q})$ have the same spectrum functor, so that they are canonically isomorphic (they fulfil the same universal property). The other statements are translations of the results in [Nes03]. $\square$

The above proposition tells us that one can translate statements about diffeological spaces purely in algebraic terms, if one works with the category $\mathrm{ALG}_{\mathcal{C}^\infty}$ or ALG_{scg}. Moreover, in this setting, one can talk of the solution space for a given smooth equation.

Definition 9. Consider the category ALG_{scg}^{op}, equipped with the Zariski topology τ_{Zar} (generated by the localization maps $A \to A[f^{-1}]$). A smoothly algebraic space is a sheaf X on the site $(\mathrm{ALG}_{scg}^{op}, \tau_{Zar})$.

Definition 10. Let $X : \mathrm{ALG}_{\mathbb{R}} \to \mathrm{SETS}$ be an algebraic space. The tangent bundle of X is defined by

$$\pi : TX(A) := X(A[\epsilon]/(\epsilon^2)) \to X(A),$$

where π is induced by the projection $A[\epsilon]/(\epsilon^2) \to A$ that sends ϵ to 0. A vector field on X is a section $\vec{v} : X \to TX$ of $\pi : TX \to X$.

Proposition 1. *Let X be a usual smooth variety, seen as an algebraic space X_a. Then a vector field on X_a identifies with a usual vector field on X.*

Proof. Recall that vector fields on X identify with derivations of $\mathcal{C}^\infty(X)$. Let $\frac{\partial}{\partial \vec{v}}$ be such a derivation. Then if A is smoothly affine, the map

$$\vec{v} : X_a(A) \to TX_a(A)$$

that sends $f : \mathcal{C}^\infty(X) \to A$ to

$$\vec{v}(f) = f + \epsilon . f \circ \frac{\partial}{\partial \vec{v}} : \mathcal{C}^\infty(X) \to A[\epsilon]/(\epsilon^2)$$

is well-defined because its image is additive and

$$\begin{array}{rcl} f(gh) + \epsilon . f(\frac{\partial (gh)}{\partial \vec{v}}) & = & f(g).f(h) + \epsilon . f(\frac{\partial g}{\partial \vec{v}}.h + g.\frac{\partial h}{\partial \vec{v}}) \\ & = & f(g).f(h) + \epsilon . f(\frac{\partial g}{\partial \vec{v}}).f(h) + f(g).f(\frac{\partial h}{\partial \vec{v}}) \\ & = & [f(g) + \epsilon . f(\frac{\partial g}{\partial \vec{v}})].[f(h) + \epsilon . f(\frac{\partial h}{\partial \vec{v}})]. \end{array}$$

Conversely, given a vector field $\vec{v} : X_a \to TX_a$, and since $\mathcal{C}^\infty(X)$ is smoothly affine by theorem 1, one can compute the value of $\vec{v}$ on the universal point

$$\mathrm{id}_X \in X_a(\mathcal{C}^\infty(X)).$$

This gives a morphism

$$\vec{v}(\mathrm{id}_X) = \mathrm{id}_X + \epsilon.D : \mathcal{C}^\infty(X) \to \mathcal{C}^\infty(X)[\epsilon]/(\epsilon^2)$$

where $D : \mathcal{C}^\infty(X) \to \mathcal{C}^\infty(X)$ is a derivation. We have thus shown the equivalence between the two notions in this case. $\square$

2.4 A simple example of differential calculus on spaces

We will now give a simple but generic example of differential calculus on spaces of fields. The same computational method works whenever the category Legos has a well-behaved notion of differential form (for example, for algebraic spaces or superspaces).

Consider the lagrangian variational problem of Newtonian mechanics, with fiber bundle

$$\pi : C = \mathbb{R}^3 \times [0,1] \to [0,1]$$

whose sections are smooth maps $x : [0,1] \to \mathbb{R}^3$, which represent a material point moving in $\mathbb{R}^3$. The space of histories is given by fixing a pair of starting and ending points for trajectories $\{x_0, x_1\}$, i.e.,

$$H = \{x \in \underline{\Gamma}(M, C),\ x(0) = x_0,\ x(1) = x_1\}.$$

More precisely, if U is a parameterizing open subset of some $\mathbb{R}^n$,

$$H(U) = \{x(t,u) \in \underline{\Gamma}(M,C)(U) \cong \mathrm{Hom}([0,1]\times U, \mathbb{R}^3),\ x(0,u) = x_0,\ x(1,u) = x_1\}.$$

If $\langle .,\rangle$ is the standard metric on $\mathbb{R}^3$ and $V : \mathbb{R}^3 \to \mathbb{R}$ is a given "potential" function, one defines [1] the action functional morphism $S : H \to \mathbb{R}$ by

$$\begin{array}{rccc} S_U : & H(U) & \to & \mathcal{C}^\infty(U) = \mathbb{R}(U) \\ & x(t,u) & \mapsto & \int_M \frac{1}{2} m \|\partial_t x(t,u)\|^2 + V(x(t,u))dt \end{array}$$

for U a parameterizing open subset of some $\mathbb{R}^n$. The differential of S is a one form ω on $\underline{\Gamma}(M,C)$, which is formalized mathematically as a family of one forms $\{x^*\omega\}$ on U for every morphism $x : U \to \underline{\Gamma}(M,C)$, which are compatible with pull-back along commutative diagrams

$$\begin{tikzcd} U \arrow[r, "x"] & \underline{\Gamma}(M,C) \\ V \arrow[u, "f"] \arrow[ur, "x'"'] & \end{tikzcd}$$

[1] We remark that for this to be well-defined, one needs to put a domination condition on $x(t,u)$ to apply Lebesgue's dominated derivation under the integral. We will implicitely add this condition everywhere in this paper.

meaning that $f^*x^*\omega = (x')^*\omega$. Concretely, if $x : M \times U \to C$ is given and $\vec{u}$ is a vector field on U, one has

$$\langle d_x S_U, \vec{u}\rangle = \frac{\partial}{\partial \vec{u}}\left[\int_M \frac{1}{2}m\|\partial_t x(t,u)\|^2 - V(x(t,u))dt\right],$$

so that

$$\langle d_x S_U, \vec{u}\rangle = \int_M \frac{1}{2}m\frac{\partial}{\partial \vec{u}}\|\partial_t x\|^2 - \frac{\partial}{\partial \vec{u}}V(x)dt$$

and one gets

$$\langle d_x S_U, \vec{u}\rangle = \int_M m\big\langle \partial_t x, \frac{\partial}{\partial \vec{u}}\partial_t x\big\rangle - \big\langle d_x V(x), \frac{\partial x}{\partial \vec{u}}\big\rangle dt.$$

By permuting the $\vec{u}$ and t derivative and integrating by parts in t, one gets

$$\langle d_x S_U, \vec{u}\rangle = \int_M \big\langle -m\partial_t^2 x - d_x V(x), \frac{\partial x}{\partial \vec{u}}\big\rangle dt + m\left[\big\langle \partial_t x, \frac{\partial x}{\partial \vec{u}}\big\rangle\right]_0^1.$$

Since $x(0,u) = x_0$ and $x(1,x) = x_1$ are constant in u, the boundary term vanishes and one gets finally

$$\langle d_x S_U, \vec{u}\rangle = \int_M \big\langle -m\partial_t^2 x - d_x V(x), \frac{\partial x}{\partial \vec{u}}\big\rangle dt.$$

The condition that $d_x S = 0$ is then equivalent to the usual Newton equation, so that the U-valued points of the space of trajectories are

$$T(U) = \{x \in H(U) | d_x S_U = 0\} = \{x \in H(U) | m\partial_t^2 x(t,u) = -V'(x(t,u))\}.$$

We remark that the above computation is completely standard in physics, and we just gave a mathematical language to formulate it. Usually, one uses functional analytic methods here, but they do not generalize properly to the super (fermionic) case, contrary to ours. Moreover, the main input of this mathematical formulation is that the spaces of histories H and trajectories T are exactly of the same nature as the spaces of parameters M and of configurations C.

2.5 Various types of spaces used in physics

The main objects of classical field theory are spaces of functions

$$\underline{\mathrm{Hom}}(X, Y)$$

between two given spaces X and Y. As explained in the previous sections, one can see these spaces as spaces similar to X and Y, if one embeds all of them in a category of sheaves on a Grothendieck site $(\textsc{Legos}, \tau)$. The choice of the site will then depend on the needs of the situation: if one needs only differential forms and X and Y are usual smooth varieties, the diffeological

setting will be sufficient and $\text{Legos} = \text{Open}_{\mathcal{C}^\infty}$. However, to have a notion of vector field on spaces, one will need a category of Legos given by some algebras, e.g. $\text{Legos} = \text{Alg}_{\mathbb{R}}^{op}$ for algebraic spaces. If one wants to work with solution spaces to smooth equations, one can also use $\text{Legos} = \text{Alg}_{scg}^{op}$.

To describe supervarieties and spaces of morphisms between them, one can not really avoid working with legos given by categories $\text{Legos} = \text{Alg}_{s,\mathbb{R}}^{op}$ opposite to superalgebras (see the forthcoming sections). This is a good reason to work from the start with algebras also in the classical smooth case.

We will also see that similar functorial and scheme theoretic methods can be applied to local functional calculus, and to the study of solution spaces to nonlinear partial differential equations. In this case, one will have $\text{Legos} = \text{Alg}_{\mathcal{D}}^{op}$, the category of $\mathcal{D}$-algebras (see the forthcoming section).

Since the resolution of an equation (or more generally of a problem) is sometimes obstructed, one will also work with more general types of space, of a homotopical nature, that encode the obstructions to the resolution of the given equations. We recall from Toen and Vezzosi's work [Toe] the idea of the construction of these spaces that allow a geometric treatment of obstruction theory.

These are essentially, in their most general form, given by some homotopy classes of functors

$$X : (\text{Legos}, \tau, W)^{op} \to (C, W_C)$$

where (Legos, W) is a model category (category with a notion of weak equivalences) equipped with the homotopical analog of a Grothendieck topology τ and (C, W_C) is a model category. The definition of the homotopy equivalence relation on these functors involves not only the week equivalences W and W_C in Legos and C but also the topology τ. We do not go into the details of their quite technical definition, but illustrate it by physical examples.

Let Δ be the category of ordered sets of the form $[1, \dots, n]$ with increasing morphisms. If C is a category, we denote C_Δ the category of functors $\Delta^{op} \to C$ and call it the category of simplicial objects in C. The first homotopical generalization of spaces one can do is to consider as building blocks a usual category Legos (for example the category $\text{Open}_{\mathcal{C}^\infty}$ of building blocks for diffeologies) and to consider simplicial presheaves

$$X : \text{Legos}^{op} \to \text{Sets}_\Delta =: \text{SSets}$$

as spaces. This gives a version of the theory of stacks, which are necessary for studying parameter spaces for objects up to isomorphisms. The main physical example of the usefulness of this setting is gauge theory: if G is a group, the space of principal G-bundles on M is to be considered as (the homotopy class of) a simplicial presheaf

$$\text{Bun}_G(M) : \text{Open}_{\mathcal{C}^\infty}^{op} \to \text{SSets}$$

whose coarse moduli space is simply the set of isomorphism classes of principal G-bundles on M parameterized by U. There is no physical reason for choosing one particular G-bundle on M and this explains why it is natural to work directly with the universal principal bundle $E = P \to \text{Bun}_G(M)$ in gauge field theory.

The obstruction theory to infinitesimal deformation of stacks can not be dealt with properly without sinking a little bit further in homotopical methods. Derived geometry is a homotopical generalization of spaces which uses as building blocks simplicial categories Legos_Δ and as spaces simplicial presheaves

$$X : \mathrm{Legos}_\Delta^{op} \to \mathrm{Sets}_\Delta =: \mathrm{SSets}.$$

These are useful for studying derived moduli spaces and necessary for making deformation and obstruction theory (for example the cotangent complex) functorial. For example, if one starts from usual scheme theory where Legos is the category of algebras, one gets as homotopical counterpart a geometry where building blocks are given by the differential graded category of differential graded (or simplicial) algebras. These kinds of space naturally appear in the quantization of gauge theory, as one can guess from the BRST-BV method for quantization (see [HT92] and the articles of Stasheff et al., for example [FLS02]).

We refer to Toen and Vezzosi's long opus [TV08b] for theory and mathematical applications of these homotopical methods. We will talk a bit along the way about their physical applications.

In all these context, given two spaces X and Y constructed from a given category of building blocks Legos, one can easily define the corresponding mapping space $\underline{\mathrm{Hom}}(X, Y)$ as the sheaf on Legos associated to the presheaf of sets

$$U \mapsto \mathrm{Hom}_{\mathrm{Legos}/U}(X \times U, Y \times U).$$

The homotopical settings need more care (resolutions) but are essentially similar.

In lagrangian relativistic physics, the space of (bosonic) Feynman history is simply the space of all sections $s : M \to E$ of a given bundle E on spacetime M. Since they are infinite dimensional, such functional spaces are not so easily studied using the observable viewpoint of physics (i.e. functions on them) because it is hard to find what is a natural and general notion of function (i.e. of an observable) on such a space. We remark also that the bundle E need not be linear (interaction particles are given by connections, which form an affine bundle) and can also be a moduli stack (as for example the moduli stack of principal bundles, which is the natural setting for gauge theory).

The punctual viewpoint is much closer to the physicists' viewpoint and allows us to consider completely canonical geometrical structures on functional spaces.

We now turn to an abstract version of geometry that allows us to treat classical spaces (bosonic) and superspaces (fermionic) with a unique and concise language.

2.6 Generalized algebras and relative geometry

So far, we have been working with the Legos categories $\mathrm{Alg}_{\mathbb{R}}^{op}$ and Alg_{scg}^{op} of usual and smoothly closed geometric algebras. The spaces used by physicists in variational calculus (for example superalgebras or $\mathcal{D}$-algebras) are based on

a generalized notion of algebra, which is usually defined as a monoid in a given symmetric monoidal (sometimes model) category.

We recall the definition of a scheme relative to a given symmetric monoidal category, due to Toen and Vaquié [TV09]. This work is of course very much inspired by Grothendieck's viewpoint of geometry. This approach to supergeometry was also emphasized in Deligne's lecture notes [DM99]. These are spaces whose building blocks are commutative monoids in the given monoidal category. They can also be generalized to the homotopical situation of symmetric monoidal model categories, as explained by Toen and Vezzosi [TV08b]. We will need these generalizations but prefer to restrict ourselves to the classical case since it is sufficiently instructive.

These methods are necessary for systematically studying spaces of fermionic-valued fields $\psi : M \to \Pi S$ from the punctual viewpoint, like for example electrons on usual spacetime, because these spaces are superspaces. They also cannot be avoided if one wants to study superalgebras geometrically (for example if M is a supermanifold, as in supersymmetric field theories), and the use of monoidal categories greatly simplifies the computations since it allows one to completely forget about signs and to work with superspaces as if these were usual spaces.

Let K be a base field of characteristic 0.

Definition 11. A symmetric monoidal category over K is a tuple $(\mathcal{C}, \otimes, \mathbb{1}, \mathrm{un}, \mathrm{as}, \mathrm{com})$ composed of

1. an abelian K-linear category $\mathcal{C}$,
2. a K-linear bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$,
3. an object $\mathbb{1}$ of $\mathcal{C}$ called the unit object,
4. for each object A of $\mathcal{C}$, two unity isomorphisms $\mathrm{un}^r_A : A \otimes \mathbb{1} \to A$ and $\mathrm{un}^l_A : \mathbb{1} \otimes A \to A$,
5. for each triple (A, B, C) of objects of $\mathcal{C}$, an associativity isomorphism
$$\mathrm{as}_{A,B,C} : A \otimes (B \otimes C) \to (A \otimes B) \otimes C,$$
6. for each pair (A, B) of objects of $\mathcal{C}$, a commutativity isomorphism
$$\mathrm{com}_{A,B} : A \otimes B \to B \otimes A,$$

that are supposed to fulfil (for more details, we refer the reader to the article on monoidal categories on wikipedia)

1. a pentagonal axiom for associativity isomorphisms,
2. a compatibility of unity and associativity isomorphisms,
3. an hexagonal axiom for compatibility between the commutativity and the associativity isomorphisms, and

4. the idempotency of the commutativity isomorphism: $\mathrm{com}_{A,B} \circ \mathrm{com}_{B,A} = \mathrm{id}_A$.

The tensor category is called closed if it has internal homomorphisms, i.e., if for every pair (B, C) of objects of $\mathcal{C}$, the functor

$$A \mapsto \mathrm{Hom}(A \otimes B, C)$$

is representable by an object $\underline{\mathrm{Hom}}(B, C)$ of $\mathcal{C}$.

The main example of a closed commutative tensor category is the category $\mathrm{V\small{ECT}}_K$ of K-vector spaces. The idea for defining differential calculus on algebras in an abstract symmetric monoidal category is to formalize it for usual algebras using only the tensor structure and morphisms in $\mathrm{V\small{ECT}}_K$.

Consider now the category whose objects are graded vector spaces

$$V = \oplus_{k \in \mathbb{Z}} V^k$$

and whose morphisms are linear maps respecting the grading. We denote it $\mathrm{V\small{ECT}}_g$. A graded vector space restricted to degree 0 and 1 is called a supervector space, and we denote $\mathrm{V\small{ECT}}_s$ the category of supervector spaces. These are abelian and even K-linear categories. If $a \in V^k$ is a homogeneous element of a graded vector space V, we denote $\deg(a) := k$ its degree. The tensor product of two graded vector spaces V and W is the usual tensor product of the underlying vector spaces equipped with the grading

$$(V \otimes W)_k = \oplus_{i+j=k} V_i \otimes W_j.$$

There is a natural homomorphism object in $\mathrm{V\small{ECT}}_g$, defined by

$$\underline{\mathrm{Hom}}(V, W) = \oplus_{n \in \mathbb{Z}} \mathrm{Hom}^n(V, W)$$

where the degree n component $\mathrm{Hom}_n(V, W)$ is the set of all linear maps $f : V \to W$ such that $f(V^k) \subset W^{k+n}$. It is an internal homomorphism object meaning that for every X, there is a natural bijection

$$\mathrm{Hom}(X, \underline{\mathrm{Hom}}(V, W)) \cong \mathrm{Hom}(X \otimes V, W).$$

The tensor product of two internal homomorphisms $f : V \to W$ and $f' : V \to W'$ is defined using the Koszul sign rule on homogeneous components. We have

$$(f \otimes g)(v \otimes w) = (-1)^{\deg(g)\deg(v)} f(v) \otimes g(w).$$

The tensor product is associative with unit $\mathbf{1} = K$ in degree 0, the usual associativity isomorphisms of K-vector spaces. The main difference with the tensor category $(\mathrm{V\small{ECT}}, \otimes)$ of usual vector space is given by the non-trivial commutativity isomorphisms

$$c_{V,W} : V \otimes W \to W \otimes V$$

defined by extending by linearity the rule

$$v \otimes w \mapsto (-1)^{\deg(v)\deg(w)} w \otimes v.$$

One thus obtains a symmetric monoidal category $(\text{Vect}_g, \otimes)$ which is moreover closed, i.e., has internal homomorphisms.

Let A be an associative unital ring. Let $(\text{Mod}_{dg}(A), \otimes)$ be the monoidal category of graded (left) A-modules, equipped with a linear map $d : C \to C$ of degree -1 such that $d^2 = 0$, with graded morphisms that commute with d and tensor product $V \otimes W$ of graded vector spaces, endowed with the differential $d : d_V \otimes \text{id}_W + \text{id}_V \otimes d_W$ (tensor product of graded maps, i.e., with a graded Leibniz rule), and also the same anticommutative commutativity constraint. This is also a closed symmetric monoidal category.

Definition 12. Let $(\mathcal{C}, \otimes)$ be a symmetric monoidal category over K. An algebra in $\mathcal{C}$ is a triple (A, μ, ν) composed of

1. an object A of $\mathcal{C}$,
2. a multiplication morphism $\mu : A \otimes A \to A$, and
3. a unit morphism $\nu : \mathbb{1} \to A$,

such that for each object V of $\mathcal{C}$, the above maps fulfil the usual associativity, commutativity and unit axiom with respect to the given associativity, commutativity and unity isomorphisms in $\mathcal{C}$. We denote $\text{Alg}_{\mathcal{C}}$ the category of algebras in $\mathcal{C}$.

In particular, a superalgebra is an algebra in the monoidal category $(\text{Vect}_s, \otimes)$. Recall that the commutativity of a superalgebra uses the commutativity isomorphism of the tensor category Vect_s, so that it actually means a graded commutativity: (A, μ) is commutative if $\mu \circ \text{com}_{A,A} = \mu$. We denote Alg_s the category of real superalgebras.

We now define, following Toen and Vaquié [TV09], the notion of scheme on $\mathcal{C}$. Given an algebra A in $\mathcal{C}$, one defines the corresponding affine scheme by its "functor of points"

$$\begin{array}{rccc} \underline{\text{Spec}}(A) : & \text{Alg}_{\mathcal{C}} & \to & \text{Sets} \\ & C & \mapsto & \underline{\text{Spec}}(A)(C) := \text{Hom}(A, C) \end{array}$$

and Yoneda's lemma implies that there is a natural bijection

$$\text{Hom}(A, B) \xrightarrow{\sim} \text{Hom}(\underline{\text{Spec}}(B), \underline{\text{Spec}}(A))$$

between algebra morphisms and affine schemes morphisms.

Definition 13. An algebra morphism $f : A \to B$ (or the corresponding morphism of functors $\underline{\text{Spec}}(B) \to \underline{\text{Spec}}(A)$) is called a standard Zariski open if it is a flat and finitely presented monomorphism:

1. (monomorphism) for every algebra C, $\underline{\mathrm{Spec}}(B)(C) \subset \underline{\mathrm{Spec}}(A)(C)$,
2. (flat) the base change functor $\_ \otimes_A B : A - \mathrm{Mod} \to B - \mathrm{Mod}$ is exact, and
3. (finitely presented) if $\underline{\mathrm{Spec}}(B/A)$ denotes $\underline{\mathrm{Spec}}(B)$ restricted to A-algebras, $\underline{\mathrm{Spec}}(B/A)$ commutes with filtered inductive limits.

A family of morphisms $\{f_i : A \to A_i\}_{i \in I}$ is called a Zariski covering if

1. for each i, $A \to A_i$ is flat,
2. there exists a finite subset $J \subset I$ such that the functor
$$\prod_{i \in J} \_ \otimes_A A_j : A - \mathrm{Mod} \to \prod_{j \in J} A_j - \mathrm{Mod}$$
is conservative, and
3. every $f_i : A \to A_i$ is Zariski open.

The Grothendieck topology generated by the Zariski coverings is called the Zariski topology. If $\underline{\mathrm{Spec}}(A) : \textsc{Alg} \to \textsc{Sets}$ is an affine scheme and $U \subset X$ is a subfunctor, U is called a Zariski open if it is the image of a morphism

$$\coprod_{i \in I} \underline{\mathrm{Spec}}(A_i) \to \underline{\mathrm{Spec}}(A)$$

induced by standard Zariski open subschemes $\underline{\mathrm{Spec}}(A_i) \subset \underline{\mathrm{Spec}}(A)$. If $X : \textsc{Alg}_{\mathcal{C}} \to \textsc{Sets}$ is any functor, a Zariski open in X is a subfunctor $U \subset X$ such that for all $\underline{\mathrm{Spec}}(A) \to X$,

$$U \times_X \underline{\mathrm{Spec}}(A) \subset \underline{\mathrm{Spec}}(A)$$

is a Zariski open morphism. We say that X is a (relative) scheme on $\mathcal{C}$ if

1. it is a sheaf for the Zariski topology, and
2. it has a covering by Zariski open subfunctors.

Using this definition, it is not hard to generalize the basic functorial results on schemes in EGA to relative schemes on $\mathcal{C}$. This also allows us to define group schemes in $\mathcal{C}$ in a completely transparent way.

2.7 Relative differential calculus

We essentially follow Lychagin [Lyc93] and also Krasilsh'chik and Verbovetsky [KV98] here. The generalization to the homotopical setting is done in Toen and Vezzosi [TV08b] for derivations.

From now on, let $\mathcal{C}$ be a symmetric monoidal category over K and (A, μ) be an algebra in $\mathcal{C}$. We will now define differential invariants of (A, μ).

A left A-module is an object M of $\mathcal{C}$ equipped with an external multiplication map $\mu_M^l : A \otimes M \to M$. If A is a commutative algebra in $\mathcal{C}$, one can put on M a right A-module structure $\mu^r : M \otimes A \to M$ defined by $\mu_M^r := \mu \circ \mathrm{com}_{M,A}$. We will implicitly use this right A-module structure in the formulas below.

Definition 14. Let M be an A-module. A morphism $D : A \to M$ in $\mathcal{C}$ is called a derivation if

$$D(fg) = fD(g) + D(f)g$$

which means more precisely that the morphism $D \circ \mu : A \otimes A \to M$ is equal to the sum $\mu_M^l \circ (\mathrm{id}_A \otimes D) + \mu_M^r \circ (D \otimes \mathrm{id}_A)$.

The description of $\mathrm{Der}(A, M)$ shows that it can be seen as a subobject $\underline{\mathrm{Der}}(A, M)$ of the internal homomorphisms $\underline{\mathrm{Hom}}(A, M)$ in $\mathcal{C}$ defined by the kernel of the morphism $\underline{\mathrm{Hom}}(A, M) \to \underline{\mathrm{Hom}}(A \otimes A, M)$ defined by

$$D \mapsto D \circ \mu - \mu_M^l \circ (\mathrm{id}_A \otimes D) + \mu_M^r \circ (D \otimes \mathrm{id}_A).$$

We remark that for $\mathcal{C} = \mathrm{V\small{ECT}}_s$ this expression can be expressed as a graded Leibniz rule by definition of the right A-module structure on M above.

The representing object for the functor $\mathrm{Der} : A - \mathrm{Mod}(\mathcal{C}) \to \mathcal{C}$ is the A-module Ω_A^1 of (Kaehler) 1-forms. One can restrict the derivation functor on A-modules to a subcategory $\mathcal{C}_g$ of $\mathcal{C}$, and this allows us to define various other types of differential form on A, called admissible differential forms for $\mathcal{C}_g$. If A is smoothly closed geometric, one usually uses geometric modules, since they give back usual differential forms in the case $A = \mathcal{C}^\infty(X)$ for X a manifold. We recall their definition from [Nes03].

Definition 15. Let A be an $\mathbb{R}$-algebra. An A-module P is called geometric if

$$\cap_{x \in \mathrm{Spec}_{\mathbb{R}}(A)} \mathfrak{m}_x M = 0,$$

where $\mathfrak{m}_x$ denotes the ideal of functions that annihilate at x, i.e., the kernel of the map $x : A \to \mathbb{R}$. We denote $\mathrm{Mod}_{g,A}$ the category of geometric modules.

Theorem 2. *Let X be a smooth variety. Admissible differential forms for the category $\mathrm{Mod}_{g,A}$ identify with usual differential forms on X. In particular, if $U \subset \mathbb{R}^n$ is a lego, there is an identification*

$$\Omega^1(U) \cong \Gamma(U, T^*U)$$

where $T^*U := U \times (\mathbb{R}^n)^*$ is the cotangent space on U.

Proof. See Nestruev [Nes03], theorem 1.43. □

We now define the differential operators on A following Lychagin in [Lyc93]. Let M and N be two left A-modules. The internal homomorphisms $\underline{\mathrm{Hom}}(M, N)$ are naturally equipped with two A-module structures

$$\mu^l : A \otimes \underline{\mathrm{Hom}}(M, N) \to \underline{\mathrm{Hom}}(M, N) \text{ and } \mu^r : \underline{\mathrm{Hom}}(M, N) \otimes A \to \underline{\mathrm{Hom}}(M, N).$$

Define the morphism $\delta^l : A \otimes \underline{\mathrm{Hom}}(M, N) \to \underline{\mathrm{Hom}}(M, N)$ by

$$\delta^l = \mu^l - \mu^r \circ \mathrm{com}_{\underline{\mathrm{Hom}}(M,N),A}$$

and define $\delta^r : \underline{\mathrm{Hom}}(M, N) \otimes A \to \underline{\mathrm{Hom}}(M, N)$ by $\delta^r = -\delta^l \circ \mathrm{com}_{A,\underline{\mathrm{Hom}}(M,N)}$. Let $\delta_n^l : A^{\otimes n} \otimes \underline{\mathrm{Hom}}(M, N) \to \underline{\mathrm{Hom}}(M, N)$ be the n-tuple composition of δ^l.

Definition 16. The module of differential operators of order n from M to N of order n is the subobject $\underline{\mathrm{Diff}}_n(M,N)$ of $\underline{\mathrm{Hom}}(M,N)$ given by intersecting the kernel of δ_n^l with $\underline{\mathrm{Hom}}(M,N)$ in $A^{\otimes n} \otimes \underline{\mathrm{Hom}}(M,N)$. If M is a fixed object of $A - \mathrm{Mod}(\mathcal{C})$, the representing object for the functor on $\underline{\mathrm{Diff}}_n(M,.) : A - \mathrm{Mod}(\mathcal{C}) \to \mathcal{C}$ is called the jet module of M, and is denoted $J^n(M)$.

In the monoidal category of usual vector spaces, this gives back the usual definition of differential operators and algebraic jet modules. To get smooth jet modules, one has to work with the subcategory of geometric modules on smoothly closed geometric algebras.

Definition 17. Let A be a superalgebra. The heart $|A|$ of A is the quotient of A by the ideal generated by the odd part A^1. One calls A smoothly closed geometric if its heart $|A|$ is smoothly closed geometric and its odd part A^1 is a geometric module over $|A|$. The corresponding category is denoted $\mathrm{A{\scriptsize LG}}_{s,scg}$.

Definition 18. The affine superspace of dimension $n|m$ with values in A is the superspace with values on a superalgebra given by

$$\mathbb{A}^{n|m}(A) = (A^0)^n \oplus (A^1)^m.$$

This superspace is affine. More precisely, one has

$$\mathbb{A}^{n|m} = \underline{\mathrm{Spec}}(\mathbb{R}[x_1, \ldots, x_n; \theta_1, \ldots, \theta_m])$$

with x_i commuting variables and θ_i anticommuting variables in the algebraic setting and

$$\mathbb{A}^{n|m} = \underline{\mathrm{Spec}}(\mathcal{C}^\infty(\mathbb{R}^n)[\theta_1, \ldots, \theta_m])$$

in the smoothly closed geometric case.

The superspaces mostly used by physicists are spaces modeled on the category $\mathrm{A{\scriptsize LG}}_{s,scg}$.

2.8 Spaces of histories and non-local observables

We now have introduced all the mathematical technology necessary to define the spaces of histories of a field theory and the notion of a non-local observable properly.

Definition 19. Let $\pi : C \to M$ be a morphism of supervarieties modeled on the category $\mathrm{L{\scriptsize EGOS}} = \mathrm{A{\scriptsize LG}}_s^{op}$ or $\mathrm{A{\scriptsize LG}}_{s,scg}^{op}$, with the topology τ_{Zar}. A space of histories for π is a subspace H of the space $\underline{\Gamma}(M,C)$ whose points with values in a superalgebra A are given by

$$\underline{\Gamma}(M,C)(A) := \Gamma_{\underline{\mathrm{Spec}}(A)}(M \times \underline{\mathrm{Spec}}(A), C \times \underline{\mathrm{Spec}}(A)).$$

A non-local observable is a morphism

$$F : H \to B$$

with values in another space B of the same type.

Abstract observables are just a weak mathematical version of what physicists call observable in DeWitt's covariant field theory [DeW03]. The space of values of a non-local observable is usually simply $B = \mathbb{R} = \mathbb{A}^{1|0}$.

Recall that M and C are defined as functors $\underline{C}, \underline{M} : \text{AFF}_{s,\mathbb{R}}^{op} \to \text{SETS}$ that are sheaves for the Zariski topology. The basic cases used to construct field theories in experimental physics are following two.

- Bosonic field theory: M is a four-dimensional (non super) variety and C is a usual fiber bundle over M (for example a connection bundle or a vector bundle). For example, the interaction particles are usually given by connections and the Higgs boson is a scalar field (usual function with $C = M \times \mathbb{R}$).

- Fermionic field theory: M is a 4-dimensional variety and C is an odd (spinorial) vector bundle over M. The algebra of "functions" on C is $\mathcal{C}^\infty(C) := \Gamma(M, \wedge C^*)$, i.e. "functions" on C that are smooth on M and antisymmetric on the fibers of $C \to M$.

In theoretical physics, there are many more possibilities. For example:

- Fermionic particles in spacetime: $M = \mathbb{R}^{0|1}$ and $C = M \times X$ with X a four-dimensional Lorentz manifold. Dirac's first quantization of the electron rests on this sound basis.

- Supersymmetric sigma models: M is a supervariety obtained by adding odd coordinates to a given classical variety $|M|$ that is usually a Riemann surface and C is of the form $M \times X$ with X, say a Calabi-Yau manifold. This is the starting point of superstring theory and of many interesting mathematical applications (Gromov-Witten theory, mirror symmetry, Topological quantum field theory).

In all cases, if A is a superalgebra, one defines the points of C and M with values in A as $\underline{C}(A) = \text{Hom}_{s\text{ALG}}(\mathcal{C}^\infty(C), A)$ and $\underline{M}(A) = \text{Hom}_{s\text{ALG}}(\mathcal{C}^\infty(M), A)$. We remark for example that for C an odd vector bundle, a function

$$f : C \to \mathbb{R} = \mathbb{A}^{1|0}$$

is simply an element of $\Gamma(M, \wedge^{2*}C)$.

In the case of a bosonic field theory, C and M being usual spaces, one can restrict $\underline{\Gamma}(M, C)$ to $\text{ALG}_{\mathbb{R}}$, ALG_{sa}, or even to $\text{Open}_{\mathcal{C}^\infty}$, depending on the needs. Concretely, the functor $\underline{\Gamma}(M, C) : \text{Open}_{\mathcal{C}^\infty} \to \text{SETS}$ sends an open subset U in some $\mathbb{R}^n$ to the families $s : U \times M \to C$ of sections of $C \to M$ parameterized by U. This is the original diffeological approach of Souriau to spaces of maps, and this approach is very close to the way physicists compute.

An $\mathbb{R}$-valued observable is then a natural transformation

$$F : \underline{\Gamma}(M, C)(.) \to \text{Hom}_{\mathcal{C}^\infty}(., \mathbb{R})$$

that sends parameterized families $s : U \times M \to C$ of sections to parameterized real numbers, i.e., to functions $r : U \to \mathbb{R}$. In particular, for $U = \{pt\}$, we associate to each section $s : M \to C$ (field) a real number $F(s) \in \mathbb{R}$.

One can also consider observable of evaluation at a point $x \in M$, which sends $s : M \times U \to C$ to $s(x,.) : U \to C$, denoted

$$\mathrm{ev}_x : \underline{\Gamma}(M, C) \to C.$$

It actually takes its values in the fiber C_x of C at x. If we suppose that C is a supervector bundle, we can make sense of the standard observables whose mean values give correlation functions, by taking the formal product $\mathrm{ev}_x\mathrm{ev}_y$ for x and y two evaluation observables. It takes values in the space algebra

$$\Lambda_{\infty,C} := \mathrm{Sym}_s^\bullet(\oplus_{x\in M} C_x),$$

where the functor Sym_s denotes the symmetric algebra taken in the super sense.

The relation of our description of observables with the "polynomial observables" of Costello [Cos10] is the following. The functorial version of Costello's observables is given by the space algebra

$$\mathcal{O}_C := \oplus_{n\geq 0}\mathrm{Hom}_{\mathbb{A}^1-\mathrm{Mod}}(\Gamma(M^n, \boxtimes^n C), \mathbb{A}^1).$$

If $A \in \mathcal{O}_C$ is an element, one defines the corresponding $\mathbb{A}^1$-valued observable

$$\begin{array}{cccc} A: & \underline{\Gamma}(M,C) & \to & \mathbb{A}^1 \\ & \varphi & \mapsto & \sum_{n\geq 0} A_n(\varphi \boxtimes \cdots \boxtimes \varphi). \end{array}$$

It can be useful to work directly with $A \in \mathcal{O}_C$ as a multilinear map

$$A : \oplus_{n\geq 0}\Gamma(M, C)^{\otimes n} \to \mathbb{A}^1,$$

where the tensor product and direct sums are made in the category of space $\mathbb{A}^1$-modules.

Another type of observable is given by local functionals, which we will study in detail in section 3: if $J^\infty(C)$ denotes the space of infinite jets of sections of $C \to M$ with coordinates (x, u, u_α) representing formal derivatives of sections, a function $L(x, u, u_\alpha) \in \mathcal{C}^\infty(J^\infty(C), \mathbb{R})$ with $|\alpha| \leq k$ defines a horizontal differential form $\omega = Ld^n x$ on $J^\infty C$. If $s : M \to C$ is a section of C, its infinite jet is a section $j_\infty s : M \to J^\infty C$ and the pull-back of ω along this section is an n form on M that can be integrated. If one fixes a compact domain $K \subset M$, one defines [2] an observable $S_{L,K}$ on sections with support in K, with values in $\mathbb{R}$, called the lagrangian action by

$$\begin{array}{cccc} S_L: & \underline{\Gamma}_K(M,C)(U) & \to & \mathcal{C}^\infty(U,\mathbb{R}) \\ & [s : U \times M \to C] & \mapsto & [u \mapsto \int_M j_\infty(s(u,.))^*\omega]. \end{array}$$

[2] Remark that this only defines a partial function, whose domain can be defined by Lebesgue's domination condition on its integrand. This condition is functorial in U and defines a subspace of the space of sections. This remark applies to all integrals written in this paper.

More generally, if $Y \subset J^\infty C$ corresponds to a differential equation (see the forthcoming section on $\mathcal{D}$-schemes), there is a natural period pairing

$$H_{k,c}(M) \times \bar{H}^k_{dR}(Y) \to \underline{\mathrm{Hom}}(\underline{Y}, \underline{\mathbb{R}})$$

(where $\underline{Y} \subset \underline{\Gamma}(M, C)$ is the subspace defined by Y) given by the same formula

$$(\alpha, \omega) \mapsto \left[\varphi(u, x) \mapsto \int_\alpha (j_\infty \varphi(u, .))^* \omega\right]$$

whose image is called the space of secondary functions on $\underline{Y}$ with values in $\mathbb{R}$.

It is in the case of fermionic field theory that one sees the real input of the functorial viewpoint of spaces of histories. Indeed, the usual notion of a point of a supervariety is not well behaved and one really has to use points valued in superalgebras. If M is usual spacetime and $C \to M$ is an odd fiber bundle, the usual duality between spaces and algebras implies that points of the space of fermionic histories $\underline{\Gamma}(M, C)$ with values in $\{pt\} = \mathrm{Spec}(\mathbb{R})$ are given by retractions $s^* : \mathcal{C}^\infty(C) \to \mathcal{C}^\infty(M)$ of the map $\pi^* : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(C)$ that induces the projection

$$\pi : \underline{C} = \mathrm{Hom}_{s\mathrm{ALG}}(\mathcal{C}^\infty(C), .) \to \mathrm{Hom}_{s\mathrm{ALG}}(\mathcal{C}^\infty(M), .) = \underline{M}.$$

Since $\mathcal{C}^\infty(C)$ is partially antisymmetric and $\mathcal{C}^\infty(M)$ is commutative, such retractions will have to be trivial on odd coordinates. If one replaces $\{pt\}$ by the superspectrum of an odd algebra $\mathrm{Spec}(A)$, the parameterized retractions $s^* : \mathcal{C}^\infty(C) \to \mathcal{C}^\infty(M) \otimes A$ can be much more general. DeWitt in his enormous book [DeW03] has chosen to use the completion of the free odd algebra on a countable number of generators

$$A = \Lambda_\infty := \widehat{\wedge^* \mathbb{R}^{(\mathbb{N})}}$$

to study fermionic fields, which is sufficient for most of the computations needed with fermionic functional integrals, but there is no physical reason to choose this algebra or another one. In any case, an $\mathbb{A}^{1,1}_{\mathbb{R}}$-valued abstract observable

$$F : \underline{\Gamma}(M, C)(.) \to \mathbb{A}^{1,1}_{\mathbb{R}}(.)$$

will associate to each retraction s^* of π^* a real supernumber parameterized by a given superalgebra A, which is the same as an element of A because $\mathbb{A}^{1,1}_{\mathbb{R}}(A) = A$. This means that a fermionic observable is essentially determined (and this is how DeWitt formalizes it) by its Λ_∞ values

$$F_{\Lambda_\infty} : \underline{\Gamma}(M, C)(\Lambda_\infty) \to \mathbb{A}^{1,1}(\Lambda_\infty) = \Lambda_\infty.$$

To convince the reader, let us give a further example of a trajectory with fermionic parameter, which is at the basis of Dirac's quantization of the electron.

Proposition 2. *Let X be a smooth variety, seen as a superspace. Denote $\pi : C = X \times \mathbb{R}^{0|1} \to \mathbb{R}^{0|1} = M$ the natural projection map. It is the configuration space for the so-called fermionic particle. There is a natural isomorphism of functors on* $\textsc{Alg}_{s,scg}$

$$\underline{\Gamma}(M, C) \cong \underline{\mathrm{Hom}}(\mathbb{R}^{0|1}, M) \cong \underline{\mathrm{Spec}}(\Omega^*(X)).$$

In particular, the real-valued functions on $\underline{\Gamma}(M, C)$, given by morphisms

$$\underline{\Gamma}(M, C) \to \mathbb{R} = \mathbb{A}^{1|0},$$

identify with even differential forms in $\Omega^{2*}(X)$.

Proof. Let A be a superalgebra. The first isomorphism follows from the trivial bundle structure of $\pi : C \to M$. The space $\underline{\mathrm{Hom}}(\mathbb{R}^{0|1}, X)$ is defined by

$$\underline{\mathrm{Hom}}(\mathbb{R}^{0|1}, X)(A) := \mathrm{Hom}_{\textsc{Alg}_{s,g}}(\mathcal{C}^\infty(X), \mathbb{R}[\theta] \otimes A)$$

where θ is an anticommuting variable. We remark that if $A = \mathbb{R}$, we get the usual set of morphisms $\mathrm{Hom}(\mathbb{R}^{0|1}, X) := \mathrm{Hom}_{\textsc{Alg}_{s,g}}(\mathcal{C}^\infty(X), \mathbb{R}[\theta])$, which identifies, since θ is odd, with $\mathrm{Hom}_{\textsc{Alg}_{s,g}}(\mathcal{C}^\infty(X), \mathbb{R}) = X$. The main advantage of adding an odd parameter algebra A is that the even part of $\mathbb{R}[\theta] \otimes A$ is $A^0 \oplus \mathbb{R}.\theta \otimes A^1$. Let $f^* : \mathcal{C}^\infty(X) \to \mathbb{R}[\theta] \otimes A$ be a morphism. Then f^* can be written as $f_0 + \theta f_1$, where $f_0 : \mathcal{C}^\infty(X) \to A^0$ is a usual morphism and $f_1 : \mathcal{C}^\infty(X) \to A^1$ is a derivation (because $\theta^2 = 0$) compatible with the $\mathcal{C}^\infty(X)$-module structure on A^1 induced by f_0 and the multiplication in A. Since A^1 is a geometric module, such a derivation can be identified with a $\mathcal{C}^\infty(X)$-module morphism $\Omega^1(X) \to A^1$. This identifies with a superalgebra morphism $\Omega^*(X) \to A$. $\square$

Let us explain why this fermionic particle is so important in physics. In Dirac's first quantization of the electron, one considers the Clifford algebra $\mathrm{Cliff}(TX, g)$ for a given lorentzian metric g on X as a quantization of the fermionic particle

$$x : \mathbb{R}^{0|1} \to X,$$

because the Clifford algebra has a filtration F such that

$$\mathrm{gr}_F^\bullet \mathrm{Cliff}(TX, g) \cong \Omega_X^*$$

and the commutator in the Clifford algebra corresponds by this isomorphism to the (odd) Poisson bracket on

$$\Omega_X^* \overset{g^{-1}}{\underset{\sim}{\to}} \wedge^* \Theta_X$$

Its state space is the space $\Gamma(M, S)$ of sections of a spinor bundle S for g, supposed to exist.

3 Local observables and differential schemes

We will now give an account of the differential calculus on local action functionals. A more general differential calculus on local functionals, called secondary differential calculus, was developed in the smooth setting by Vinogradov [Vin01]. It is essentially a homotopical version of the following.

3.1 Partial differential equations and $\mathcal{D}$-algebras

Many spaces of functions used in physics are described by partial differential equations. To study spaces of solutions of partial differential equations from a functorial viewpoint, one needs to know what kind of algebraic structure is necessary to write down a given partial differential equation.

Let $\pi : C = \mathbb{R} \times \mathbb{R} \to \mathbb{R} = M$ be the trivial bundle. A polynomial partial differential equation on the sections of this bundle is a polynomial expression

$$F(t, x, \partial_t x, \dots, \partial_t^n x) = 0$$

that involves the parameter $t \in M$, a section $x : M \to C$ and its derivatives. To write down the same expression in a more general algebraic structure, one needs:

- an $\mathcal{O}_M = \mathbb{R}[t]$-algebra A,
- with a compatible action of ∂_t, and
- a morphism $\mathcal{O}_C = \mathbb{R}[t, x] \to A$.

One can also see this datum as a $\mathcal{D}_M$-module A (where $\mathcal{D}_M$ is the algebra of differential operators on M, generated by the action of ∂_t and $\mathcal{O}_M$ on $\mathcal{O}_M$), equipped with a multiplication

$$\mu : A \otimes_{\mathcal{O}_M} A \to A$$

that is $\mathcal{D}_M$-linear, for the $\mathcal{D}_M$-module structure on the tensor product given by Leibniz's rule. One moreover needs a morphism

$$\mathcal{O}_C \to A$$

to make sense of $x \in A$. If such a datum is given, one writes the solution space to $F(t, \partial_t^i x) = 0$ with values in A as

$$\mathrm{Sol}_{F=0}(A) := \{x \in A, F(t, \partial_t^i x) = 0\}.$$

One sees here a strong similarity with the space of solutions of a polynomial equation, described in 2.3. The point is that the equation F itself lives in the universal $\mathcal{O}_C \otimes_{\mathcal{O}_M} \mathcal{D}_M$-algebra, that is the jet $\mathcal{D}_M$-algebra $\mathrm{Jet}(\mathcal{O}_C) = \mathbb{R}[t, x_i]$ with action of ∂_t given by $\partial_t x_i = x_{i+1}$.

So one gets a perfect analogy between polynomials and polynomial partial differential equations given by

Equation	Polynomial	Partial differential
Formula	$P(x)=0$	$F(t,\partial^\alpha x)=0$
Naive variable	$x\in\mathbb{R}$	$x\in\mathrm{Hom}(\mathbb{R},\mathbb{R})$
Algebraic structure	commutative unitary ring A	$\mathcal{D}_M$-algebra A
Free structure	$P\in\mathbb{R}[x]$	$F\in\mathrm{Jet}(\mathcal{O}_C)$
Solution space	$\{x\in A, F(x)=0\}$	$\{x\in A, F(t,\partial^\alpha x)=0\}$

To work with non-polynomial smooth partial differential equations, one has to work in the category $\textsc{Alg}_{scg}$ of smoothly closed geometric algebras. In this setting, the jet algebra is the smooth geometric closure of the polynomial jet algebra and the equation F lives in

$$\mathcal{C}^\infty(J^\infty C) := \overline{\mathrm{Jet}(\mathcal{O}_C)}^{scg}.$$

3.2 $\mathcal{D}$-modules

We recall some properties of the categories of $\mathcal{D}$-modules that can mostly be found in [KS90], [Kas03] and [Sch94] for most of them, except the compound tensor structure, which was defined in [BD04].

Let M be a smooth variety of dimension n and $\mathcal{D}$ be the algebra of differential operators on M. We recall that, locally on M, one can write an operator $P\in\mathcal{D}$ as a finite sum

$$P=\sum_\alpha a_\alpha\partial^\alpha$$

with $a_\alpha\in\mathcal{O}_M$,

$$\partial=(\partial_1,\dots,\partial_n):\mathcal{O}_M\to\mathcal{O}_M^n$$

the universal derivation and α some multi-indices.

To write down the equation $Pf=0$ with f in an $\mathcal{O}_M$-module $\mathcal{S}$, one needs to define the universal derivation $\partial:\mathcal{S}\to\mathcal{S}^n$. This is equivalent to giving $\mathcal{S}$ the structure of a $\mathcal{D}$-module. The solution space of the equation with values in $\mathcal{S}$ is then given by

$$\mathrm{Sol}_P(\mathcal{S}) := \{f\in\mathcal{S},\ Pf=0\}.$$

We remark that

$$\mathrm{Sol}_P:\mathrm{Mod}(\mathcal{D})\to\textsc{Vect}_{\mathbb{R}_M}$$

is a functor that one can think of as representing the space of solutions of P. Denote $\mathcal{M}_P$ the cokernel of the $\mathcal{D}$-linear map

$$\mathcal{D}\xrightarrow{.P}\mathcal{D}$$

given by right multiplication by P. Applying the functor $\mathcal{H}om_{\mathcal{M}(\mathcal{D})}(.,\mathcal{S})$ to the exact sequence

$$\mathcal{D}\xrightarrow{.P}\mathcal{D}\longrightarrow\mathcal{M}_P\to 0,$$

we get the exact sequence

$$0\to\mathcal{H}om_{\mathrm{Mod}(\mathcal{D})}(\mathcal{M}_P,\mathcal{S})\to\mathcal{S}\xrightarrow{P.}\mathcal{S},$$

which gives a natural isomorphism

$$\mathrm{Sol}_P(\mathcal{S}) = \mathcal{H}om_{\mathrm{Mod}(\mathcal{D})}(\mathcal{M}_P, \mathcal{S}).$$

This means that the $\mathcal{D}$-module $\mathcal{M}_P$ represents the solution space of P, so that $\mathcal{D}$-modules are a convenient setting for the functor of point approach to linear partial differential equations.

We remark that it is even better to consider the derived solution space

$$\mathbb{R}\mathrm{Sol}_P(\mathcal{S}) := \mathbb{R}\mathcal{H}om_{\mathrm{Mod}(\mathcal{D})}(\mathcal{M}_P, \mathcal{S})$$

because it also encodes information on the inhomogeneous equation

$$Pf = g.$$

Recall that the subalgebra $\mathcal{D}$ of $\mathrm{End}_{\mathbb{R}}(\mathcal{O})$ is generated by left multiplication by functions in $\mathcal{O}_M$ and by the derivation induced by vector fields in Θ_M. There is a natural right action of $\mathcal{D}$ on the $\mathcal{O}$-module Ω^n_M by

$$\omega.\partial = -L_\partial \omega$$

with L_∂ the Lie derivative.

There is a tensor product in the category $\mathrm{Mod}(\mathcal{D})$ given by

$$\mathcal{M} \otimes \mathcal{N} := \mathcal{M} \otimes_{\mathcal{O}} \mathcal{N}$$

where the $\mathcal{D}$-module structure on the tensor product is given on vector fields $\partial \in \Theta_M$ by Leibniz's rule

$$\partial(m \otimes n) = (\partial m) \otimes n + m \otimes (\partial n).$$

There is also an internal homomorphism $\mathcal{H}om(\mathcal{M}, \mathcal{N})$ given by the $\mathcal{O}$-module $\mathcal{H}om_{\mathcal{O}}(\mathcal{M}, \mathcal{N})$ equipped with the action of derivations $\partial \in \Theta_M$ by

$$\partial(f)(m) = \partial(f(m)) - f(\partial m).$$

The functor

$$\mathcal{M} \mapsto \mathcal{M}^r := \Omega^n_M \otimes_{\mathcal{O}} \mathcal{M}$$

induces an equivalence of categories between the categories $\mathrm{Mod}(\mathcal{D})$ and $\mathrm{Mod}(\mathcal{D}^{op})$ of left and right $\mathcal{D}$-modules whose quasi-inverse is

$$\mathcal{N} \mapsto \mathcal{N}^\ell := \mathcal{H}om_{\mathcal{O}_M}(\Omega^n_M, \mathcal{N}).$$

Definition 20. Let $\mathcal{S}$ be a right $\mathcal{D}$-module. The de Rham functor with values in $\mathcal{S}$ is the functor

$$\mathrm{DR}_{\mathcal{S}} : \mathrm{Mod}(\mathcal{D}) \to \mathrm{VECT}_{\mathbb{R}_M}$$

that sends a left $\mathcal{D}$-module to

$$\mathrm{DR}_{\mathcal{S}}(\mathcal{M}) := \mathcal{S} \overset{\mathbb{L}}{\otimes}_{\mathcal{D}} \mathcal{M}.$$

The de Rham functor with values in $\mathcal{S} = \Omega^n_M$ is denoted DR and simply called the de Rham functor. One also denotes $\mathrm{DR}^r_{\mathcal{S}}(\mathcal{M}) = \mathcal{M} \overset{\mathbb{L}}{\otimes}_{\mathcal{D}} \mathcal{S}$ if $\mathcal{S}$ is a fixed left $\mathcal{D}$-module and $\mathcal{M}$ is a varying right $\mathcal{D}$-module, and $\mathrm{DR}^r := \mathrm{DR}^r_{\mathcal{O}}$.

Proposition 3. *The natural map*

$$\begin{array}{ccc} \Omega^n_M \otimes_{\mathcal{O}} \mathcal{D} & \to & \Omega^n_M \\ \omega \otimes Q & \mapsto & \omega Q \end{array}$$

extends to a $\mathcal{D}^{op}$-linear quasi-isomorphism

$$\Omega^*_M \otimes_{\mathcal{O}} \mathcal{D}[n] \xrightarrow{\sim} \Omega^n_M.$$

We will see that in the super setting, this proposition can be taken as a definition of the right $\mathcal{D}$-modules of volume forms, the so called Berezinian.

Proposition 4. *Let $\mathcal{S}$ be a right coherent $\mathcal{D}$-module and $\mathcal{M}$ be a coherent left $\mathcal{D}$-module. There is a natural quasi-isomorphism*

$$\mathbb{R}\mathrm{Sol}_{\mathbb{D}(\mathcal{M})}(\mathcal{S}) := \mathbb{R}\mathrm{Hom}(\mathbb{D}(\mathcal{M}), \mathcal{S}) \cong \mathrm{DR}_{\mathcal{S}}(\mathcal{M}),$$

where $\mathbb{D}(\mathcal{M}) := \mathbb{R}\mathrm{Hom}(\mathcal{M}, \mathcal{D})$ is the $\mathcal{D}$-module dual of $\mathcal{M}$.

3.3 $\mathcal{D}$-modules on supervarieties and the Berezinian

We refer to Penkov's article [Pen83] for a complete study of the Berezinian in the $\mathcal{D}$-module setting.

Let M be a supervariety of dimension $n|m$. As explained in subsection 2.7 one defines Ω^1_M as the representing object for the internal derivation functor $\underline{Der}(\mathcal{O}_M, .)$ on geometric $\mathcal{O}_M$-modules. One also defines Ω^*_M as the superexterior power

$$\Omega^*_M := \wedge^* \Omega^1_M.$$

The super version of Proposition 3 can be taken as a definition of the Berezinian, as a complex of $\mathcal{D}$-modules, up to quasi-isomorphism.

Definition 21. The Berezinian of M is defined in the derived category of $\mathcal{D}_M$-modules by the formula

$$\mathrm{Ber}_M := \Omega^*_M \otimes_{\mathcal{O}} \mathcal{D}[n].$$

The complex of integral forms $I_{*,M}$ is defined by

$$I_{*,M} := \mathbb{R}\mathrm{Hom}_{\mathcal{D}}(\mathrm{Ber}_M, \mathrm{Ber}_M).$$

The following proposition (see [Pen83], 1.6.3) gives a description of the Berezinian as a $\mathcal{D}$-module.

Proposition 5. *The Berezinian complex is concentrated in degree* 0 *and equal there to*

$$\mathrm{Ber}_M := \mathcal{E}xt^n_{\mathcal{D}}(\mathcal{O}, \mathcal{D}).$$

The functor

$$\begin{array}{ccc} \mathrm{Mod}(\mathcal{D}) & \to & \mathrm{Mod}(\mathcal{D}^{op}) \\ \mathcal{M} & \mapsto & \mathcal{M}^r := \mathrm{Ber}_M \otimes \mathcal{M} \end{array}$$

is an equivalence of categories with quasi-inverse $\mathcal{M} \mapsto \mathcal{M}^\ell := \mathrm{Ber}_M^{-1} \otimes \mathcal{M}$. This equivalence and the tensor product of left $\mathcal{D}$-modules over $\mathcal{O}$ induce a monoidal structure on $\mathrm{Mod}(\mathcal{D}^{op})$, denoted $\otimes^!$.

In the supersetting, the equivalence of left and right $\mathcal{D}$-modules, given by the functor

$$\mathcal{M} \mapsto \mathrm{Ber}_M \otimes_{\mathcal{O}} \mathcal{M}$$

of twist by the Berezinian right $\mathcal{D}$-module, can be computed by using the definition

$$\mathrm{Ber}_M := \Omega_M^* \otimes_{\mathcal{O}} \mathcal{D}[n]$$

and passing to degree 0 cohomology.

A more explicit description of the complex of integral forms (up to quasi-isomorphism) is given by

$$I_{*,M} := \mathbb{R}\mathcal{H}om_{\mathcal{D}}(\mathrm{Ber}_M, \mathrm{Ber}_M) \cong \mathcal{H}om_{\mathcal{D}}(\Omega_M^* \otimes_{\mathcal{O}} \mathcal{D}[n], \mathrm{Ber}_M)$$

so that we get

$$I_{*,M} \cong \mathcal{H}om_{\mathcal{O}}(\Omega_M^*[n], \mathrm{Ber}_M) \cong \mathcal{H}om_{\mathcal{O}}(\Omega_M^*[n], \mathcal{O}) \otimes_{\mathcal{O}} \mathrm{Ber}_M$$

and in particular $I_{n,M} \cong \mathrm{Ber}_M$.

We remark that proposition 3 shows that if M is a usual variety, then Ber_M is quasi-isomorphic with Ω_M^n, and this implies that

$$I_{*,M} \cong \mathcal{H}om_{\mathcal{O}}(\Omega_M^*[n], \mathcal{O}) \otimes_{\mathcal{O}} \mathrm{Ber}_M \cong \wedge^* \Theta_M \otimes_{\mathcal{O}} \Omega_M^n \xrightarrow{i} \Omega_M^*,$$

where i is the insertion morphism. This implies the isomorphism

$$I_{n-p,M} \cong \Omega_M^p,$$

so that in the purely even case, integral forms essentially identify with usual differential forms.

We recall from Bernstein and Leites' work [BL77] that, with a convenient notion of compactly supported homology $H_{*,c}(M)$ on a given supermanifold M, there is an integration pairing

$$H_{*,c}(M) \times h^*(I_{*,M}) \to \mathbb{R}$$

that reduces to the usual integration pairing

$$H_{*,c}(M) \times H_{dR}^*(M) \to \mathbb{R}$$

in the classical case. The integration of an integral form in I_{n-m-p} is done on subsupermanifolds of dimension $p|m$ of a given supermanifold of dimension $n|m$.

3.4 $\mathcal{D}$-algebras and partial differential equations

In this subsection, we will work with varieties modeled on the category $\textsc{Alg}_{\mathbb{R}}$, $\textsc{Alg}_{scg}$, $\textsc{Alg}_s$ or $\textsc{Alg}_{s,scg}$.

The tensor structure on the category of left $\mathcal{D}$-modules allows us to define $\mathcal{D}$-algebras and $\mathcal{D}$-schemes using the philosophy of relative geometry in monoidal categories, like in section 2.6.

We recall that the $\mathcal{D}$-module structure on $\mathcal{M}\otimes_{\mathcal{O}}\mathcal{N}$ is given by Leibniz's rule

$$D(a\otimes b) = Da\otimes b + a\otimes Db$$

on the level of derivations. This means that the following notion of $\mathcal{D}$-algebra is just a $\mathcal{D}$-module equipped with a multiplication that fulfils Leibniz rule for derivations.

Definition 22. A (commutative) $\mathcal{D}$-algebra is a monoid in the monoidal category of (left) $\mathcal{D}$-module, i.e., it is a $\mathcal{D}$-module $\mathcal{A}$ equipped with a multiplication morphism of $\mathcal{D}$-modules

$$\mu : \mathcal{A}\otimes_{\mathcal{O}_X}\mathcal{A}\to\mathcal{A}$$

and a unit map $1:\mathcal{O}_X\to\mathcal{A}$ that fulfil associativity, unity and commutativity axioms. If $\mathcal{A}$ is a $\mathcal{D}$-algebra, a module over $\mathcal{A}$ is given by a $\mathcal{D}$-module $\mathcal{M}$ and a morphism $\mu:\mathcal{A}\otimes\mathcal{M}\to\mathcal{M}$ of external multiplication that is compatible with unit and multiplication on $\mathcal{A}$ in the usual sense.

Definition 23. A $\mathcal{D}$-space is a sheaf on the site $(\mathrm{A}{\scriptstyle\mathrm{LG}}_{\mathcal{D}}^{op},\tau_{Zar})$ of $\mathcal{D}$-algebras with their Zariski topology.

We now introduce the differential algebraic analog of polynomial algebra, called jet algebra, by recalling the following result [BD04], 2.3.2.

Proposition 6. *Let $\pi : C\to M$ be a smooth map between varieties. There exists a free $\mathcal{D}_M$-algebra generated by $\mathcal{O}_C$, denoted* $\mathrm{Jet}(\mathcal{O}_C)$*. More precisely, one has, for every $\mathcal{D}$-algebra $\mathcal{A}$, a natural isomorphism*

$$\mathrm{Hom}_{\mathcal{D}_M-\mathrm{A}{\scriptstyle\mathrm{LG}}}(\mathrm{Jet}(\mathcal{O}_C),\mathcal{A})\cong\mathrm{Hom}_{\mathcal{O}_M-\mathrm{A}{\scriptstyle\mathrm{LG}}}(\mathcal{O}_C,\mathcal{A}).$$

Its spectrum is denoted $\mathrm{Jet}(\pi)$*, or simply* $\mathrm{Jet}(C)$*.*

Proof. The algebra $\mathrm{Jet}(\mathcal{O}_C)$ is given by the quotient of the symmetric algebra

$$\mathrm{Sym}^\bullet(\mathcal{D}\otimes_{\mathcal{O}_X}\mathcal{O}_C)$$

by the ideal generated by the elements $\partial(1\otimes r_1.1\otimes r_2 - 1\otimes r_1r_2)\in\mathrm{Sym}^2(\mathcal{D}\otimes_{\mathcal{O}_X} R)+\mathcal{D}\otimes_{\mathcal{O}_X}R$ and $\partial(1\otimes 1_R-1)\in\mathcal{D}\otimes R+\mathcal{O}_X$, $r_i\in R$, $\partial\in\mathcal{D}$, 1_R the unit of R. □

We remark that the Jet algebra is not in general finitely generated as an algebra over $\mathcal{O}_M$, but by definition, it is finitely generated as a $\mathcal{D}_M$-algebra if $\mathcal{O}_C$ is finitely generated over $\mathcal{O}_M$. If $s:M\to C$ is a section, we denote $j_\infty s : M\to\mathrm{Jet}(C)$ the corresponding map with values in the jet space.

We have defined here only the algebraic jet space, but one can define the usual jet space by working with (super)algebras that are smoothly closed geometric. Indeed, the smooth closure of the algebraic jet algebra gives the algebra

of functions on usual infinite jet space, by construction. Our methods thus apply also to the smooth case, if one works with the convenient category of algebras and modules over them as in [Nes03].

We now give a definition of a partial differential equation and of its spaces of solutions, which works equally well in the smooth, algebraic or supergeometric setting.

Definition 24. A partial differential equation on the sections of $\pi : C \to M$ is a $\mathcal{D}$-ideal $\mathcal{I}$ in $\mathrm{Jet}(\mathcal{O}_C)$. Its differential solution space is the $\mathcal{D}$-space whose points with values in $\mathrm{Jet}(\mathcal{O}_C)$-$\mathcal{D}$-algebras $\mathcal{A}$ are

$$\mathrm{Sol}_{\mathcal{D},\mathcal{I}=0}(\mathcal{A}) = \{x \in \mathcal{A},\ f(x) = 0,\ \forall f \in \mathcal{I}\}.$$

Its solution space is the subspace of $\underline{\Gamma}(M, C)$ whose points with values in a test algebra A (in $\textsc{Alg}_{scg}$, $\textsc{Alg}_{\mathbb{R}}$ or $\textsc{Alg}_{s,scg}$) are given by

$$\mathrm{Sol}_{\mathcal{I}=0}(A) = \{s(t, u) \in \underline{\Gamma}(M, C)(A),\ f \circ (j_{\infty,t}s)(t, u) = 0,\ \forall f \in \mathcal{I}\},$$

where we identify $\underline{\Gamma}(M, C)(A)$ with a subset $\mathrm{Hom}(M \times \underline{\mathrm{Spec}}(A), C)$.

We remark that one has an isomorphism of $\mathcal{D}$-spaces

$$\mathrm{Sol}_{\mathcal{D},\mathcal{I}=0} \cong \underline{\mathrm{Spec}}_{\mathcal{D}}(\mathrm{Jet}(\mathcal{O}_C)/\mathcal{I})$$

which means that the differential solution space is in some sense (which will be clarified in the next subsection) finite dimensional. This is very different of the diffeological solution space that is far away from being a finite-dimensional manifold in general. This finite dimensionality can be seen as the mathematical reason why physicist like to work with local functionals.

3.5 Local functionals and local differential calculus

Let M be a supermanifold of dimension $n|m$ and $\mathcal{M}$ be a $\mathcal{D}_M$-module. We suppose that the underlying manifold $|M|$ is oriented.

Definition 25. The central de Rham cohomology of $\mathcal{M}$ is defined by

$$h(\mathcal{M}) := \mathrm{Ber}_M \otimes_{\mathcal{D}} \mathcal{M}.$$

The variational de Rham complex of $\mathcal{M}$ is defined by

$$\mathrm{DR}_{var}(\mathcal{M}) := (I_{n-m-*,M} \otimes_{\mathcal{O}} \mathcal{D}) \otimes_{\mathcal{D}} \mathcal{M}.$$

We remark that in the classical case of dimension $n|0$, the variational de Rham complex identifies with the usual de Rham complex

$$\mathrm{DR}(\mathcal{M}) := \Omega^n_M \overset{\mathbb{L}}{\otimes}_{\mathcal{D}} \mathcal{M} = (\Omega^*_M \otimes_{\mathcal{O}} \mathcal{D}[n]) \otimes_{\mathcal{D}} \mathcal{M}$$

and the central de Rham cohomology is isomorphic to

$$h(\mathcal{M}) = \Omega^n_M \otimes_{\mathcal{D}} \mathcal{M}.$$

Let $\pi : C \to M$ be a bundle and $H \subset \underline{\Gamma}(M, C)$ be a subspace that is a solution space of a given partial differential $\mathcal{I}_H$ on $\underline{\Gamma}(M, C)$. Let $\mathcal{A} = \mathrm{Jet}(\mathcal{O}_C)/\mathcal{I}_H$ be the corresponding $\mathcal{D}$-algebra. We suppose that it is $\mathcal{D}$-smooth.

Integral forms can be integrated on subsupermanifolds. This allows us to define [3] an integration pairing.

Proposition 7. *There is a well-defined integration pairing*

$$\begin{array}{ccc} H_{*,c}(M) \times h^*(\mathrm{DR}_{var}(\mathcal{A})) & \to & \underline{\mathrm{Hom}}(H, \mathbb{R}) \\ (\Sigma, \omega) & \mapsto & F_{\Sigma,\omega} : s(t,u) \mapsto \int_\Sigma (j_{\infty,t} s(t,u))^* \omega. \end{array}$$

Proof. This follows from the fact that the integral of a total derivative is zero. □

Definition 26. A functional $F_{\Sigma,\omega} : H \to \mathbb{R}$ in the image of the above pairing is called a local functional on H.

Let $d : \mathcal{A} \to \Omega^1_{\mathcal{A}}$ be the universal derivation with values in $\mathcal{A}[\mathcal{D}]$-modules ($\mathcal{A}$-modules in the tensor category of $\mathcal{D}$-modules). Let $\Omega^*_{\mathcal{A}} = \wedge^*_{\mathcal{A}} \Omega^1_{\mathcal{A}}$ be the corresponding algebra of differential forms. One can generalize the above notion of local functional to differential forms.

Proposition 8. *There is a well-defined integration pairing*

$$\begin{array}{ccc} H_{*,c}(M) \times h^*(\mathrm{DR}_{var}(\Omega^k_{\mathcal{A}})) & \to & \Omega^k_H \\ (\Sigma, \omega) & \mapsto & \nu_{\Sigma,\omega} : [s : M \times U \to C] \mapsto \int_\Sigma (j_\infty s)^* \omega \in \Omega^k_U. \end{array}$$

Definition 27. A differential form $\nu_{\Sigma,\omega} \in \Omega^k_H$ in the image of the above pairing is called a local differential form on H.

Definition 28. Let $\mathcal{A}$ be a smooth $\mathcal{D}$-algebra (see [BD04], chapter 2). The $\mathcal{A}^r[\mathcal{D}^{op}]$-module of local vector fields on $\mathcal{A}$ is defined as the $\mathcal{A}[\mathcal{D}]$-dual of differential forms, i.e., by the formula

$$\Theta_{\mathcal{A}} := \mathcal{H}om_{\mathcal{A}[\mathcal{D}]}(\Omega^1_{\mathcal{A}}, \mathcal{A}[\mathcal{D}]).$$

The finite dimensionality of the $\mathcal{D}$-space of solutions of a partial differential equation can be explained by the following proposition from [BD04], chapter 2.

Proposition 9. *Let $\mathcal{B} = \mathrm{Jet}(\mathcal{O}_C)$ and $p : \mathrm{Jet}(\mathcal{O}_C) \to C$ be the projection map. The natural map*

$$\mathcal{B}[\mathcal{D}] \otimes_{\mathcal{B}} p^* \Omega^1_{C/M} \to \Omega^1_{\mathcal{B}}$$

is an isomorphism in the jet case. The rank of $\Omega^1_{\mathcal{B}}$ as a $\mathcal{B}[\mathcal{D}]$-module is equal to the rank of $\Omega^1_{C/M}$ as an $\mathcal{O}_C$-module.

[3] The functionals in play have a domain of definition given by Lebesgue's domination condition on the integrand.

Corollary 1. *Let $\mathcal{B} = \mathrm{Jet}(\mathcal{O}_C)$, $\mathcal{B}^r := \mathrm{Ber}_M \otimes_{\mathcal{O}_M} \mathcal{B}$ and $p : \mathrm{Jet}(C) \to C$ be the projection map. The natural morphism of $\mathcal{B}^r[\mathcal{D}^{op}]$-modules*

$$\Theta_{\mathcal{B}} \to (p^*\Theta_{C/M}) \otimes_{\mathcal{B}^r} \mathcal{B}^r[\mathcal{D}^{op}]$$

is an isomorphism. The rank of $\Theta_{\mathcal{B}}$ as a $\mathcal{B}^r[\mathcal{D}^{op}]$-module is equal to the rank of $\Theta_{C/M}$ as an $\mathcal{O}_C$-module.

Proof. The morphism $p : \mathrm{Jet}(C) \to C$ induces an exact sequence of $\mathcal{B}$-modules

$$0 \to p^*\Omega^1_{C/M} \to \Omega^1_{\mathrm{Jet}(C)/M} \to \Omega^1_{\mathrm{Jet}(C)/C} \to 0$$

and a natural morphism

$$\mathcal{D} \otimes_{\mathcal{O}} p^*\Omega^1_{C/M} \to \Omega^1_{\mathrm{Jet}(C)/M}$$

of left $\mathcal{D}$-modules (the right-hand side is equipped with its canonical $\mathcal{D}$-module structure). This gives a natural map

$$\Theta_{\mathcal{B}} := \mathcal{H}om_{\mathcal{B}[\mathcal{D}]}(\Omega^1_{\mathrm{Jet}(C)/M}, \mathcal{B}[\mathcal{D}]) \to \mathcal{H}om_{\mathcal{B}[\mathcal{D}]}(\mathcal{D} \otimes_{\mathcal{O}} p^*\Omega^1_{C/M}, \mathcal{B}[\mathcal{D}])$$

and combining it with the natural isomorphisms

$$\begin{array}{rcl} \mathcal{H}om_{\mathcal{B}[\mathcal{D}]}(\mathcal{D} \otimes_{\mathcal{O}} p^*\Omega^1_{C/M}, \mathcal{B}[\mathcal{D}]) & \cong & \mathcal{H}om_{\mathcal{B}}(p^*\Omega^1_{C/M}, \mathcal{B}[\mathcal{D}]) \\ & \cong & \mathcal{H}om_{\mathcal{B}}(p^*\Omega^1_{C/M}, \mathcal{B}) \otimes \mathcal{D} \\ & =: & (p^*\Theta_{C/M}) \otimes \mathcal{D} \end{array}$$

induces a natural map

$$\Theta_{\mathcal{B}} \to (p^*\Theta_{C/M}) \otimes \mathcal{D}^{op}.$$

The fact that it is an isomorphism comes from the fact that the natural map

$$\mathcal{B}[\mathcal{D}] \otimes_{\mathcal{B}} p^*\Omega^1_{C/M} \to \Omega^1_{\mathcal{B}}$$

is an isomorphism in the jet case. ∎

3.6 Variational calculus

We now recall our general notion of variation problem. We use here superspaces modeled on geometric superalgebras.

Definition 29. A lagrangian variational problem is composed of the following data:

1. a space M called the parameter space for trajectories,
2. a space C called the configuration space for trajectories,
3. a morphism $\pi : C \to M$ (often supposed to be surjective),

4. a subspace $H \subset \underline{\Gamma}(M, C)$ of the space of sections of π

$$\underline{\Gamma}(M, C) := \{x : M \to C,\ \pi \circ x = \mathrm{id}\},$$

called the space of histories, and

5. a functional $S : H \to \mathbb{R}$ called the action functional.

The space of classical trajectories for the variational problem is the subspace T of H defined by

$$T = \{x \in H |\ d_x S = 0\}.$$

If B is another space, a classical B-valued observable is a functional $F : T \to B$ and a quantum B-valued observable is a functional $F : H \to B$.

Virtually every example of variational calculus that can be found in the classical physical literature is of the following type.

Definition 30. A variational problem is called local if the following hold.

1. The space of histories $H \subset \underline{\Gamma}(M, C)$ is defined by a differential equation $\mathcal{I}_H \subset \mathrm{Jet}(\mathcal{O}_C)$, such that $\mathrm{Jet}(\mathcal{O}_C)/\mathcal{I}_H$ is $\mathcal{D}$-smooth,
2. The action functional is the local functional associated to a cohomology class $S \in h(\mathcal{A})$ for $\mathcal{A} = \mathrm{Jet}(\mathcal{O}_C)/\mathcal{I}_H$.

Suppose that we are given a local variational problem. We suppose that $\mathcal{A}$ is $\mathcal{D}$-smooth (see [BD04]). Using the biduality isomorphism

$$\Omega^1_{\mathcal{A}} \cong \mathcal{H}om_{\mathcal{A}^r[\mathcal{D}^{op}]}(\Theta_{\mathcal{A}}, \mathcal{A}^r[\mathcal{D}^{op}]),$$

one gets a well-defined isomorphism

$$h(\Omega^1_{\mathcal{A}}) \cong \mathcal{H}om_{\mathcal{A}[\mathcal{D}]}(\mathrm{Ber}_M^{-1} \otimes \Theta_{\mathcal{A}}, \mathcal{A}).$$

To the given action functional $S \in h(\mathcal{A})$ corresponds its differential $h(d)(S) \in h(\Omega^1_{\mathcal{A}})$ and by the above isomorphism, an insertion map

$$i_{dS} : \mathrm{Ber}_M^{-1} \otimes \Theta_{\mathcal{A}} \to \mathcal{A}.$$

Definition 31. The image of the above insertion map is called the Euler-Lagrange ideal and denoted $\mathcal{I}_S$. The lagrangian variational problem is said to have simplifying histories if its space of trajectories $T = \{x \in H,\ d_x S = 0\}$ identifies with the solution space of the Euler-Lagrange ideal, i.e.,

$$T \cong \mathrm{Sol}_{\mathcal{I}_S = 0} \subset H.$$

To sum up, a variational problem has simplifying histories if the conditions imposed on trajectories to define H annihilate the boundary terms of the integration by part that is used to compute $d_x S$ explicitely (see for example section 2.4).

4 Gauge theories and homotopical geometry

In this last section, we briefly describe a coordinate-free formulation of gauge theory and of the classical BV formalism using the language of $\mathcal{D}$-schemes of [BD04]. We are inspired here by a huge physical literature, starting with [HT92] and [FH90] as general references, but also [Sta97] and [Sta98]for some homotopical inspiration, and [FLS02], [Bar10] and [CF01] for explicit computations.

For the consistency of this article, we want to insist on the geometrical meaning of these constructions, continuing to deal with spaces defined by their functor of points. We will thus use without further comment

- the language of homotopical geometry, referring to [Toe] for a survey and more references, and
- the language of pseudo-tensor operations, which we will call here local operations, referring to [BD04], chapter 1 and 2, for their definition and use.

Just recall from Chapter 2 of [BD04] the following definition (we replace everywhere the word pseudo-tensor in this reference by the word local).

Definition 32. Let $\mathcal{A}$ be a $\mathcal{D}$-algebra, $\mathcal{M}$ be an $\mathcal{A}[\mathcal{D}]$-module and $\mathcal{A}^r := \mathrm{Ber}_M \otimes \mathcal{A}$ be the corresponding algebra in the symmetric monoidal category $(\mathrm{Mod}(\mathcal{D}^{op}), \otimes^!)$ of right $\mathcal{D}$-modules with $\mathcal{M}^r := \mathrm{Ber}_M \otimes \mathcal{M}$. A local 2-ary operation on the $\mathcal{A}[\mathcal{D}]$-module $\mathcal{M}$ is a morphism

$$\mathcal{M}^r \boxtimes \mathcal{M}^r \to \Delta_* \mathcal{M}^r$$

where $\Delta : M \to M \times M$ is the diagonal embedding. The inner dual of a projective $\mathcal{A}[\mathcal{D}]$-module $\mathcal{M}$ of finite rank is the $\mathcal{A}[\mathcal{D}]$-module defined by

$$\mathcal{M}^\circ := \mathrm{Ber}_M^{-1} \otimes \mathcal{H}om_{\mathcal{A}[\mathcal{D}]}(\mathcal{M}, \mathcal{A}[\mathcal{D}]).$$

If $\mathcal{M}$ is an $\mathcal{A}^r[\mathcal{D}^{op}]$-module, we denote

$$\mathcal{M}^\ell := \mathrm{Ber}_M^{-1} \otimes \mathcal{M}$$

the corresponding $\mathcal{A}[\mathcal{D}]$-module.

Definition 33. A variational problem $(\pi : C \to M, H, S \in h(\mathcal{A}))$ with simplifying histories is called a gauge theory. The kernel of its insertion map

$$i_{dS} : \Theta_{\mathcal{A}}^\ell \to \mathcal{A}$$

is called the space $\mathcal{N}_S$ of Noether identities. Its right version

$$\mathcal{N}_S^r = \mathrm{Ber}_M \otimes \mathcal{N}_S \subset \Theta_{\mathcal{A}}$$

is called the space of Noether gauge symmetries.

We remark that there is a natural local Lie bracket operation

$$[.,.] : \Theta_{\mathcal{A}} \boxtimes \Theta_{\mathcal{A}} \to \Delta_* \Theta_{\mathcal{A}}$$

that plays the role of the Lie bracket between local vector fields. We refer to Beilinson and Drinfeld's book [BD04] for the following proposition.

Proposition 10. *The local Lie bracket of vector fields extends naturally to an odd local Poisson bracket on the dg-$\mathcal{A}$-algebra*

$$\mathcal{A}_P := \mathrm{Sym}_{dg}([\Theta^{\ell}_{\mathcal{A}}[1] \xrightarrow{i_{dS}} \mathcal{A}]).$$

One can see the dg-algebra $\mathcal{A}_P$ as a dg-$\mathcal{D}$-space

$$\begin{array}{cccc} P := \mathrm{Spec}(\mathcal{A}_P): & dg - \mathcal{A} - \textsc{Alg} & \to & \textsc{SSets} \\ & \mathcal{R} & \mapsto & s\mathrm{Hom}_{dg-Alg_{\mathcal{D}}}(\mathcal{A}_P, \mathcal{R}). \end{array}$$

One has then by construction that

$$\pi_0(P) \cong \mathrm{Sol}_{\mathcal{I}_S = 0},$$

i.e., the classical (non-homotopical part of) P is exactly the $\mathcal{D}$-space of critical points of the action functional S. However, it can have non-trivial higher homotopy, if the space of Noether identities is non-trivial.

Definition 34. The above space P is called the non-proper derived critical space of the given system.

Corollary 2. *The natural map*

$$\mathcal{N}^r_S \boxtimes \mathcal{N}^r_S \to \Delta_* \Theta_{\mathcal{A}}$$

induced by the bracket on local vector fields always factors through $\Delta_* \mathcal{N}^r_S$ and the natural map

$$\mathcal{N}^r_S \boxtimes \mathcal{A}^r / \mathcal{I}^r_S \to \Delta_* \mathcal{A}^r / \mathcal{I}^r_S$$

is a Lie $\mathcal{A}$-algebroid action.

Let $\mathfrak{g}_S \to \mathcal{N}_S$ be a projective $\mathcal{A}[\mathcal{D}]$-resolution of the Noether identities, and suppose (to simplify, but this is rarely the case) that the dg-algebra

$$\mathcal{B} = \mathrm{Sym}_{dg}([\mathfrak{g}_S[2] \to \Theta^{\ell}_{\mathcal{A}}[1] \to \mathcal{A}])$$

is a cofibrant resolution of $\mathcal{A}/\mathcal{I}_S$, whose differential is denoted d_{KT}. From the point of view of derived geometry, differential forms on this resolution give a definition of the cotangent complex on the $\mathcal{D}$-space $\underline{\mathrm{Spec}}_{\mathcal{D}}(\mathcal{A}/\mathcal{I}_S)$ of critical points of the action functional. One can think of the derived $\mathcal{D}$-stack

$$\begin{array}{cccc} \mathbb{R}T := \mathbb{R}\underline{\mathrm{Spec}}_{\mathcal{D}}(\mathcal{A}/\mathcal{I}_S): & dg - \textsc{Alg}_{\mathcal{D}} & \to & \textsc{SSets} \\ & \mathcal{R} & \mapsto & s\mathrm{Hom}_{dg-\textsc{Alg}_{\mathcal{D}}}(\mathcal{B}, \mathcal{R}) \end{array}$$

as a proper solution space for the Euler-Lagrange equation (physicist's language), or a proper derived critical space. It is a homotopical replacement of the $\mathcal{D}$-space $\underline{\mathrm{Spec}}_{\mathcal{D}}(\mathcal{A}/\mathcal{I}_S)$.

For the following definition, we give a local version of the notion of L_∞-algebroid, whose classical definition can be found in Loday and Vallette's book [LV10]. Roughly speeking, a local L_∞-algebroid is a representation of the L_∞-operad in the pseudo-tensor category of $\mathcal{A}^r[\mathcal{D}^{op}]$-modules. This can be shown to be equivalent to the datum of an inner coderivation on some coalgebra. Since we are mostly interested in the Chevalley-Eilenberg complex, we will use this definition.

Definition 35. A local L_∞-algebroid structure on a graded $\mathcal{A}[\mathcal{D}]$-module $\mathcal{L}$ is given by an $\mathcal{A}^r[\mathcal{D}^{op}]$-inner coderivation d of degree 1 of the cocommutative coassociative inner coalgebra $(C^c(\mathcal{L}^r[1]),\Delta)$ over $\mathcal{A}^r[\mathcal{D}^{op}]$, where $C^c(\mathcal{L}^r[1])$ is the inner exterior algebra of $\mathcal{L}^r$ equipped with its natural coalgebra structure $\Delta : C^c(\mathcal{L}^r[1]) \to C^c(\mathcal{L}^r[1]) \wedge C^c(\mathcal{L}^r[1])$ defined by

$$\Delta(\gamma_1) = 0$$

and

$$\Delta(\gamma_1\wedge\cdots\wedge\gamma_n) = \frac{1}{2}\sum_{k=0}^{n-1}\frac{1}{k!(n-k)!}\sum_{\epsilon\in S_n}\mathrm{sgn}\ (\epsilon).\gamma_{\epsilon(1)}\wedge\cdots\wedge\gamma_{\epsilon(k)}\bigwedge\gamma_{\epsilon(k+1)}\wedge\cdots\wedge\gamma_{\epsilon(n)}.$$

If we suppose that all components of $\mathcal{L}^r[1]$ are projective of finite type, we can dualize d to a derivation d° on $\mathrm{Sym}(\mathcal{L}^\circ[1])$ (where $\mathcal{L}^\circ$ is the inner dual of $\mathcal{L}$). This gives a definition of the Chevalley-Eilenberg complex of the $\mathcal{A}[\mathcal{D}]$-L_∞-algebroid $\mathcal{L}$

$$(C(\mathcal{L}), d_{CE}) := (\mathrm{Sym}(\mathcal{L}^\circ[1]), d^\circ).$$

Proposition 11. *Let $\bar{\mathfrak{g}}_S \to \mathcal{N}_S/\mathcal{I}_S$ be a projective resolution of the space $\mathcal{N}/\mathcal{I}_S$ of on-shell Noether identities. There is a natural local L_∞-algebroid structure on $\bar{\mathfrak{g}}_S$. If $\bar{\mathfrak{g}}_S$ is bounded finitely generated, there is a well-defined Chevalley-Eilenberg differential*

$$d_{CE} : \mathrm{Sym}(\bar{\mathfrak{g}}_S^\circ[1]) \to \mathrm{Sym}(\bar{\mathfrak{g}}_S^\circ[1]).$$

Proof. The result follows from the pseudo-tensor version of theorem 3.5 of [BM03], by homotopical transfer of the local Lie bracket on $\mathcal{N}_S/\mathcal{I}_S$ to a local L_∞-structure on $\mathfrak{g}_S/\mathcal{I}_S$. $\square$

The aim of the Batalin-Vilkoviski formalism is to define a Poisson differential graded algebra $(\mathcal{A}_{BV}, D)$ whose derived $\mathcal{D}$-stack

$$\begin{array}{cccc} \mathbb{R}\underline{\mathrm{Spec}}_{\mathcal{D}}(\mathcal{A}_{BV}, D): & dg-\mathrm{ALG}_{\mathcal{D}} & \to & \mathrm{SSETS} \\ & \mathcal{B} & \mapsto & s\mathrm{Hom}_{dg-\mathrm{ALG}_{\mathcal{D}}}((\mathcal{A}_{BV}, D), \mathcal{B}) \end{array}$$

is a kind of homotopical space of leaves

$$\mathbb{R}\mathrm{Spec}(\mathcal{A}/\mathcal{I}_S)\underset{\mathbb{L}}{/}(\mathcal{N}_S^r/\mathcal{I}_S^r)$$

of the derived critical space $\mathbb{R}\mathrm{Spec}(\mathcal{A}/\mathcal{I}_S)$ by the "foliation by gauge orbits" defined by the Lie algebroid $\mathcal{N}_S^r/\mathcal{I}_S^r$ of on-shell gauge symmetries. The differential D is essentially obtained, under additional hypothesis, by combining in a neat way the above Chevalley-Eilenberg differential d_{CE} for the $\mathfrak{g}_S/\mathcal{I}_S$-module $\mathcal{A}/\mathcal{I}_S$ with the Koszul-Tate differential d_{KT} on the cofibrant resolution $\mathcal{B}$ of $\mathcal{A}/\mathcal{I}_S$. This neat combination could be done, for example if $\mathfrak{g}_S \to \mathcal{N}_S$ was a projective resolution of the Noether identities, by extending the local action map

$$\mathcal{N}_S^r \boxtimes \mathcal{A}^r/\mathcal{I}_S^r \to \Delta_*\mathcal{A}^r/\mathcal{I}_S^r$$

to an ∞-action

$$\mathfrak{g}_S^r \boxtimes \mathcal{B}^r \to \Delta_*\mathcal{B}^r$$

of the local L_∞-$\mathcal{A}$-algebroid $\mathfrak{g}_S$ on the resolution $\mathcal{B}$ of $\mathcal{A}/\mathcal{I}_S$, and taking the total complex of the associated Chevalley-Eilenberg complex (see [BD04], section 1.4.5)

$$(C(\mathfrak{g}_S, \mathcal{B}), d_{CE}).$$

Remark however that this object is only an $\mathbb{R}_M$-algebra and one would like to have an $\mathcal{A}$-algebra here, by replacing the Chevalley-Eilenberg complex by an *inner* version of it. The existence of the inner Chevalley-Eilenberg complex is only given under very strong finite-dimension conditions, that are not fulfilled in the above construction. The essentially role of the Batalin-Vilkovisky construction is to give a systematic way to fill the above conceptual gap, by using smaller generating spaces of Noether symmetries.

Indeed, remark that the left hand side of the natural map

$$\wedge^2\Theta_{\mathcal{A}}^\ell \to \mathcal{N}.$$

is not a finitely generated $\mathcal{A}[\mathcal{D}]$-module (contrary to what would occur in a finite dimensional geometric situation) for the same reason that $\mathcal{D} \otimes_{\mathcal{O}} \mathcal{D}$ is not $\mathcal{D}$-coherent. This shows that it is hard to find an $\mathcal{A}[\mathcal{D}]$-finitely generated off-shell projective resolution $\mathfrak{g}_S$ of the space $\mathcal{N}_S$ of Noether identities. Another problem is that such a projective resolution does not give, in general, a cofibrant resolution of the $\mathcal{A}$-algebra $\mathcal{A}/\mathcal{I}_S$ because the differential graded symmetric algebra functor $\mathrm{Sym}_{dg-\mathcal{A}}$ is not always exact. This motivates the following construction, that is also useful for computational purposes.

Generating spaces of Noether gauge symmetries can be defined by adapting Tate's construction [Tat57] to the local context. We are inspired here by Stasheff's paper [Sta97].

Definition 36. A generating space of Noether identities is a tuple $(\mathfrak{g}_S, \mathcal{A}_n, i_n)$ composed of

1. a negatively graded projective $\mathcal{A}[\mathcal{D}]$-module $\mathfrak{g}_S$,
2. a negatively indexed family $\mathcal{A}_n$ of dg-$\mathcal{A}$-algebras with $\mathcal{A}_0 = \mathcal{A}$, and
3. for each $n \leq -1$, an $\mathcal{A}[\mathcal{D}]$-linear morphism $i_n : \mathfrak{g}_S^{n+1} \to Z^n\mathcal{A}_n$ to the n-cycles of $\mathcal{A}_n$,

such that if one extends $\mathfrak{g}_S$ by setting $\mathfrak{g}_S^1 = \Theta_{\mathcal{A}}^{\ell}$ and if one sets

$$i_0 = i_{dS} : \Theta_{\mathcal{A}}^{\ell} \to \mathcal{A},$$

1. one has for all $n \leq 0$ an equality

$$\mathcal{A}_{n-1} = \mathrm{Sym}_{\mathcal{A}_n}([\mathfrak{g}_S^{n+1}[-n+1] \otimes_{\mathcal{A}} \mathcal{A}_n \overset{i_n}{\to} \underset{0}{\mathcal{A}_n}]),$$

2. the natural projection map

$$\mathcal{A}_{KT} := \varinjlim \mathcal{A}_n \to \mathcal{A}/\mathcal{I}_S$$

   is a cofibrant resolution, called the Koszul-Tate algebra, whose differential is denoted d_{KT}.

Lemma 2. *The complex*

$$\mathcal{P}_{KT} := [\mathcal{A}_{KT}/(\Theta_{\mathcal{A}}^{\ell})]^{\leq -2}$$

of components of degree smaller than -2 in the quotient algebra of the Koszul-Tate algebra by the ideal of local vector fields maps to the space $\mathcal{N}_S$ of Noether identities, and surjects onto $\mathcal{N}_S/\mathcal{I}_S$. Its underlying graded module is

$$\mathcal{P}_{KT} = \mathrm{Sym}_g(\mathfrak{g}_S[2]).$$

The inclusion $\mathfrak{g}_S[2] \subset \mathcal{P}_{KT}$ induces a degree 1 map

$$\widetilde{d} : \mathfrak{g}_S[1] \to \mathfrak{g}_S.$$

Proof. The first statements follow from the definition of the Koszul-Tate algebra. The inclusion and projection of homogeneous components induce natural maps

$$\mathfrak{g}_S \to \mathcal{P}_{KT} \text{ and } \mathcal{P}_{KT} \to \mathfrak{g}_S,$$

that can be composed with the differential on $\mathcal{P}_{KT}$ to define $\widetilde{d}$. □

Definition 37. One says that a generating space of Noether identities is

1. strongly regular if the graded space $\mathfrak{g}_S$ is bounded with finitely generated projective components,
2. on-shell complete if the natural map $\mathfrak{g}_S \to \mathcal{N}_S$ induces a projective resolution

$$\mathfrak{g}_S/\mathcal{I}_S \to \mathcal{N}_S/\mathcal{I}_S$$

   of $\mathcal{N}_S/\mathcal{I}_S$ as an $\mathcal{A}/\mathcal{I}_S$-module, with differential induced by $\widetilde{d}$.
3. on-shell algebraically complete if the natural map $\mathcal{P}_{KT} \to \mathcal{N}_S$ induces a projective resolution

$$\mathcal{P}_{KT}/\mathcal{I}_S \to \mathcal{N}_S/\mathcal{I}_S.$$

Since $\mathcal{D} \otimes_{\mathcal{O}} \mathcal{D}$ is usually not $\mathcal{D}$-coherent, the higher degree homogeneous components of the Koszul-Tate algebra (e.g, the component $\wedge^2\Theta^\ell_{\mathcal{A}}$, that corresponds to trivial Noether identities) are usually not of finite type over $\mathcal{A}[\mathcal{D}]$, even under the strongly regular hypothesis. This problem is specific to local field theory and does not occur in finite dimensional geometry: the graded algebras in play are of finite type, but their higher homogeneous components are not finitely generated modules.

Proposition 12. *Let $\mathfrak{g}_S$ be an on-shell complete (respectively, on-shell algebraically complete) generating space of Noether identities. There is a natural local L_∞-algebroid structure on $\bar{\mathfrak{g}}_S := \mathfrak{g}_S/\mathcal{I}_S$ (resp. $\mathcal{P}_{KT}/\mathcal{I}_S$). If $\mathfrak{g}_S$ is on-shell complete and strongly regular, there is a well-defined degree 1 map*

$$d_{CE} : \mathrm{Sym}(\mathfrak{g}_S^\circ[1]) \to \mathrm{Sym}(\mathfrak{g}_S^\circ[1])$$

induced by the L_∞-algebroid structure on $\bar{\mathfrak{g}}_S$.

Proof. The condition of on-shell completeness means that the natural map

$$\bar{\mathfrak{g}}_S \to \mathcal{N}_S/\mathcal{I}_S$$

is a projective resolution. The result follows from the pseudo-tensor version of theorem 3.5 of [BM03], by homotopical transfer of the local Lie bracket on $\mathcal{N}_S/\mathcal{I}_S$ to a local L_∞-structure on $\bar{\mathfrak{g}}_S$. The extension of d_{CE} to $\mathrm{Sym}(\mathfrak{g}_S^\circ[1])$ is given by the fact that $\mathfrak{g}_S^\circ \to \mathfrak{g}_S^\circ/\mathcal{I}_S$ is componentwise surjective and $\mathfrak{g}_S^\circ$ has projective components. $\square$

Definition 38. One says that the gauge symmetries close off-shell if there exists a generating space of Noether gauge symmetries $\mathfrak{g}_S$ that is finitely $\mathcal{A}[\mathcal{D}]$-generated, and whose image $\bar{\mathfrak{g}}_S$ in $\Theta^\ell_{\mathcal{A}}$ admits a bracket

$$\bar{\mathfrak{g}}_S^r \boxtimes \bar{\mathfrak{g}}_S^r \to \Delta_*\bar{\mathfrak{g}}_S^r$$

induced by the local bracket on vector fields. One says that the theory is N-reducible is there exists a minimal strongly regular generating space of length N. In particular, 0-reducible gauge theories are called irreducible gauge theories.

Definition 39. Let $\mathfrak{g}_S$ be a strongly regular generating space of the Noether gauge symmetries. Such a generating space is called a space of antighosts of the gauge theory. The inner dual space (that is well-defined because of the strong regularity hypothesis)

$$\mathfrak{g}_S^\circ := \mathrm{Ber}_M^{-1} \otimes \mathcal{H}om_{\mathcal{A}[\mathcal{D}]}(\mathfrak{g}_S, \mathcal{A}[\mathcal{D}])$$

is called the space of ghosts.

Theorem 3. *Let $\mathfrak{g}_S$ be a strongly regular generating space of Noether identities and $\mathfrak{g}_S^\circ$ its inner dual. The bigraded algebra*

$$\mathcal{A}_{BV,bigrad} := \mathrm{Sym}_{bigrad}\left(\begin{bmatrix} \mathfrak{g}_S[2] & \oplus & \mathrm{Ber}_M^{-1}\otimes\Theta_{\mathcal{A}}[1] & \oplus & \mathcal{A} \\ & & & & \oplus \\ & & & & {}^t\mathfrak{g}_S^\circ[-1] \end{bmatrix}\right),$$

where ${}^t\mathfrak{g}_S^\circ$ is the vertical chain graded space associated to $\mathfrak{g}_S^\circ$, is equiped with a natural local bracket

$$\{.,.\} : \mathcal{A}^r_{BV,bigrad} \boxtimes \mathcal{A}^r_{BV,bigrad} \to \Delta_* \mathcal{A}^r_{BV,bigrad}$$

called the antibracket.

Proof. There is a natural duality pairing

$$\langle .,. \rangle : \mathfrak{g}_S^r \boxtimes (\mathfrak{g}_S^\circ)^r \to \Delta_* \mathcal{A}^r$$

between antighosts and ghosts. Similarly to the finite-dimensional case treated in [KS87], this duality and the isomorphism

$$\mathrm{gr}^\bullet \mathrm{Cliff}_{inner}(\mathfrak{g}_S^r[2] \oplus (\mathfrak{g}_S^\circ)^r[-1], \langle .,. \rangle) \cong \mathrm{Sym}(\mathfrak{g}_S^r[2] \oplus (\mathfrak{g}_S^\circ)^r[-1])$$

induce a local Poisson bracket on

$$\mathrm{Sym}(\mathfrak{g}_S[2] \oplus \mathfrak{g}_S^\circ[-1]) \cong \mathrm{Sym}(\mathfrak{g}_S[2]) \otimes \mathrm{Sym}(\mathfrak{g}_S^\circ[-1]).$$

Combining this with the local Schouten-Nijenhuis bracket

$$\{.,.\} : (\wedge^* \Theta_\mathcal{A}) \boxtimes (\wedge^* \Theta_\mathcal{A}) \to \Delta_*(\wedge^* \Theta_\mathcal{A}),$$

one gets a local Poisson bracket on the bigraded algebra

$$\mathcal{A}_{BV,bigrad} := \mathrm{Sym}_{bigrad}\left(\left[\begin{array}{ccccc} \mathfrak{g}_S[2] & \oplus & \mathrm{Ber}_M^{-1} \otimes \Theta_\mathcal{A}[1] & \oplus & \mathcal{A} \\ & & & & \oplus \\ & & & & {}^t\mathfrak{g}_S^\circ[-1] \end{array}\right]\right).$$

□

Definition 40. Let $\mathfrak{g}_S$ be a strongly regular generating space of Noether identities. The corresponding BV algebra is the local Poisson algebra $\mathcal{A}_{BV,bigrad}$. A solution to the classical master equation is an $S_{cm} \in h(\mathcal{A}_{BV,bigrad})$ such that

1. the degree $(0,0)$ component of S_{cm} is S,
2. a component of S_{cm}, denoted S_{KT}, induces the Koszul-Tate differential $d_{KT} = \{S_{KT}, .\}$ on antifields of degrees $(k, 0)$, and
3. the master equation
$$\{S_{cm}, S_{cm}\} = 0$$
(meaning $D^2 = 0$ for $D = \{S_{cm}, .\}$) is fulfilled in $h(\mathcal{A}_{BV,bigrad})$.

One can also add some conditions on S_{cm} related to the on-shell Chevalley-Eilenberg differential d_{CE}.

The main theorem of homological perturbation theory, given in a physical language in [HT92], chapter 17 (DeWitt indices), can be formulated in our case by the following.

Theorem 4. *Let $\mathfrak{g}_S$ be a strongly regular and on-shell complete generating space of Noether symmetries. There exists a solution to the corresponding classical master equation.*

As explained above, the space $\mathbb{R}\underline{\mathrm{Spec}}_{\mathcal{D}}(\mathcal{A}_{BV}, D)$ can be thought as a kind of homotopical space of leaves

$$\mathbb{R}\mathrm{Spec}(\mathcal{A}/\mathcal{I}_S)\underset{\mathbb{L}}{/}\mathcal{N}_S^r$$

of the foliation induced by the Noether gauge symmetries $\mathcal{N}_S^r$ on the derived critical space $\mathbb{R}\underline{\mathrm{Spec}}_{\mathcal{D}}(\mathcal{A}/\mathcal{I}_S)$. It is naturally equipped with a homotopical Poisson structure, which gives a nice starting point for quantization.

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